Midterm Stochastic Modeling (400646)

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October 25, 2017, 12:00 - 14:00 hours

This (midterm) exam consist of four exercises. The use of books or a graphical calculator is not allowed.

At each part it is indicated between square brackets how many points can be achieved for the corresponding part. The grade for the midterm is given by $p/3 + 1$ where p is the total number of points earned by you. Please include your name and student number on all papers and motivate all your answers clearly.

Good luck!

Exercise 1

Consider a discrete-time Markov chain with state space $\{1, 2, 3, 4, 5, 6\}$ and matrix of transition probabilities

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 2/3 & 0 & 1/3 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 1/3 & 0 & 0 & 2/3 & 0 \end{pmatrix}$$

(a) [2 pt.] Draw the state diagram of possible transitions and determine the classes of communicating states. Also, determine $\mathbb{P}(X_n = 1 \mid X_0 = 1)$ for $n = 1, 2, 3, 4, 5$.

(b) [3 pt.] Does the limiting distribution exist? And the occupancy distribution? If they exist, determine the limiting and/or the occupancy distribution.

(c) [2 pt.] Which transition should be added such that the Markov chain becomes irreducible?

Exercise 2

A team of consultants occasionally carries out data science projects. In order to be productive, they save up a couple of projects and then work for one day on such a batch of projects. Specifically, at the end of each day the team checks how many projects are open (waiting to be carried out). If at least 3 projects are open at the end of the day, the team works on all of those projects during the next day. It is assumed that they are able to complete the entire batch of projects in exactly 1 day. When they work on data science projects, there are no new requests for projects arriving. When the team does not work on data science project during a day, then either 0, 1, or 2 new projects arrive with probabilities $1/3$, $1/2$, and $1/6$, respectively.

(a) [3 pt.] Formulate an appropriate discrete-time Markov chain to analyze the number of open data science projects for the team. Specify the state diagram and transition probabilities. Also give the balance equations.

(b) [2 pt.] What is the expected number of days until the team works on data science projects again when they just finished a batch of projects?

It turns out that working for one day on data science projects is not always sufficient to complete a batch of data science projects. With probability p, the team needs exactly one day, and with probability $1-p$ the team needs exactly two days (during which also no new projects arrive).

(c) [2 pt.] Formulate an appropriate Markov chain and specify the matrix of one-step transition probabilities.

Exercise 3

Consider a system with two parallel servers. Customers arrive according to a Poisson process with rate 2. Arriving customers go to server 1 with probability $1/3$ and to server 2 with probability $2/3$, independent of other customers. The service times of server i are exponentially distributed with rate μ_i, $i = 1, 2$.

(a) [2 pt.] What is the probability that during two time units (that is, an interval of length 2) no customer arrives at server 1? And what is the joint probability that during two time units no customer arrives at server 1 and at least 2 customers arrive at server 2?

Access to server 1 may be blocked. With probability $3/4$ server 1 remains available, whereas with probability $1/4$ access to server 1 is blocked during the next two time units. When access to server 1 is blocked, arriving customers always go to server 2.

(b) [2 pt.] What is the probability that during two time units exactly 1 customer arrives at server 2?

(c) [2 pt.] Suppose that customer A is being served by server 1 and customer B is served by server 2. What is the probability that A has completed service before B? And what is the probability that A has completed service at least t time units before B?

Exercise 4

For the transport of live animals, a company has two vehicles available. Customers place a request for live animal transport according to a Poisson process with rate 2. When both vehicles are occupied, customers go elsewhere. The time for transport is assumed to follow an exponential distribution with rate 4.

(a) [3 pt.] Formulate an appropriate continuous-time Markov chain to analyze the number of occupied vehicles. Let p_i be the limiting probability that i vehicles are occupied. Show that $p_0 = 8/13$, $p_1 = 4/13$, and $p_2 = 1/13$.

(b) [2 pt.] Suppose that the company uses the following cost structure. When a vehicle is idle (no transport), the company calculates k per time unit per idle vehicle. When both vehicles are occupied, no new requests can be accepted. The company calculates s per time unit that both vehicles are occupied. Determine the long-run average costs per time unit.

The company decides to improve the cleaning process after transport. When a vehicle has completed its transport of live animals, it is being cleaned, after which it is available for transport again. The time for cleaning is assumed to follow an exponential distribution with rate ν.

(c) [2 pt.] Formulate an appropriate continuous-time Markov chain to analyze the availability of vehicles. Specify the state diagram and transition rates.