Midterm Stochastic Modeling (400646) - Solutions

The solutions are always provisionary

Vrije Universiteit Amsterdam
Faculty of Sciences, Department of Mathematics

October 25, 2017, 12:00 - 14:00 hours

Exercise 1

(a) [2 pt.] See Figure 1 for the transition diagram. The classes of communicating states are $\{1, 2, 3\}$ (absorbing), $\{4\}$, and $\{5, 6\}$ (transient).

Moreover, $\mathbb{P}(X_n = 1 \mid X_0 = 1) = 0$ for $n = 1, 2$ and $\mathbb{P}(X_n = 1 \mid X_0 = 1) = \frac{2}{3}\left(\frac{1}{3}\right)^{n-3}$ for $n = 3, 4, 5$.

(b) [3 pt.] Both the limiting distribution and occupancy distribution exist, since the DTMC is aperiodic, positive recurrent (finite state space), and there is one absorbing class. Also, note that $\pi_i = \hat{\pi}_i$ and $\pi_4 = \pi_5 = \pi_6 = 0$. The remaining probabilities can be found using the balance equations

$$\pi_1 = \frac{2}{3}\pi_3, \quad \pi_2 = \pi_1, \quad \pi_3 = \pi_2 + \frac{1}{3}\pi_3.$$

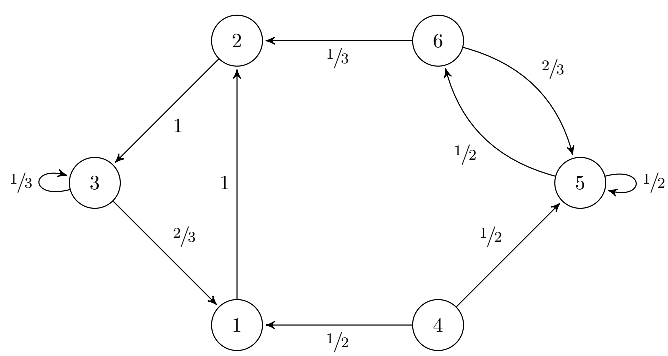

Figure 1: State diagram of the DTMC of exercise 1a.

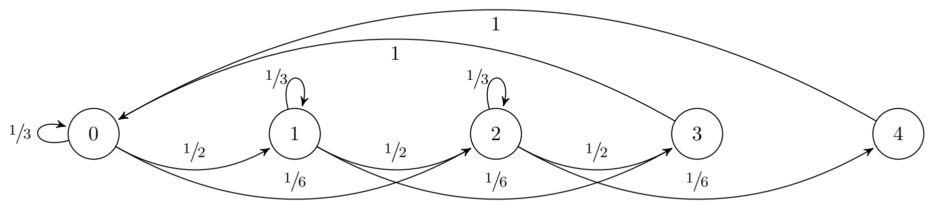

Figure 2: State diagram of the DTMC of exercise 2a.

E.g. expressing everything in terms of π_3 and using normalization gives the equation $\pi_3\left(\frac{2}{3}+\frac{2}{3}+1\right)=1$, yielding the final result

$$\pi=\hat{\pi}=\left(\frac{2}{7},\frac{2}{7},\frac{3}{7},0,0,0\right).$$

(c) [2 pt.] The simplest option is to add a transition from one of the states $1, 2, 3$ to state 4.

Exercise 2

(a) [3 pt.] Let X_n be the number of open data science projects at the end of day n. Then $\{X_n, n=0,1,\dots\}$ is a discrete-time Markov chain (DTMC) on the state space $I=\{0,1,\dots,4\}$. The state diagram with transition probabilities is given in Figure 2. The balance equations are:

$$\begin{aligned}
\pi_0 &= \frac{1}{3}\pi_0+\pi_3+\pi_4\\
\pi_1 &= \frac{1}{3}\pi_1+\frac{1}{2}\pi_0\\
\pi_2 &= \frac{1}{3}\pi_2+\frac{1}{2}\pi_1+\frac{1}{6}\pi_0\\
\pi_3 &= \frac{1}{2}\pi_2+\frac{1}{6}\pi_1\\
\pi_4 &= \frac{1}{6}\pi_2
\end{aligned}$$

(b) [2 pt.] Define m_i as the expected number of days to hit the set $\{3,4\}$ given that the DTMC is now in state i. We then have

$$\begin{aligned}
m_0 &= 1 + \frac{1}{3}m_0 + \frac{1}{2}m_1 + \frac{1}{6}m_2 \\
m_1 &= 1 + \frac{1}{3}m_1 + \frac{1}{2}m_2 \\
m_2 &= 1 + \frac{1}{3}m_2
\end{aligned}$$

Solving 'backward', we first obtain $m_2 = \frac{3}{2}$, then $m_1 = \frac{21}{8}$, and finally the desired result $m_0 = \frac{123}{32}$.

(c) [2 pt.] Extend the state space by adding a state $*$ denoting that the team works for the second consecutive day on a data science project (at day n). Then $\{X_n, n = 0, 1, \ldots\}$ is a DTMC on state space $\{0, 1, \ldots, 4, *\}$. The one-step transition probability matrix reads

$$\mathbf{P} = \begin{pmatrix}
1/3 & 1/2 & 1/6 & 0 & 0 & 0 \\
0 & 1/3 & 1/2 & 1/6 & 0 & 0 \\
0 & 0 & 1/3 & 1/2 & 1/6 & 0 \\
p & 0 & 0 & 0 & 0 & 1-p \\
p & 0 & 0 & 0 & 0 & 1-p \\
1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Exercise 3

Define $N_i(t)$ as the number of arriving customers for server i during $[0, t]$, $i = 1, 2$. Due to thinning, it holds that $N_1(t)$ PP($2/3$) and $N_2(t)$ PP($4/3$).

(a) [2 pt.] From the above, the probability that no customer arrives at server 1 during $[0, t]$ is

$$\mathbb{P}(N_1(2) = 0) = e^{-2\times 2/3} = e^{-4/3}.$$

Due to independence, we also have

$$\begin{aligned}
\mathbb{P}(N_1(2) = 0; N_2 \geq 2) &= \mathbb{P}(N_1(2) = 0)\,(1 - \mathbb{P}(N_2(2) \leq 1)) \\
&= e^{-4/3}\left(1 - e^{-8/3}(1 + 8/3)\right) \\
&= e^{-4/3} - e^{-4}\frac{11}{3}.
\end{aligned}$$

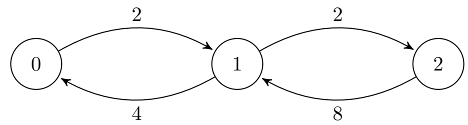

Figure 3: State diagram of the CTMC of exercise 4a.

(b) [2 pt.] Let $N(t)$ be the total number of arriving customers during $[0, t]$. Conditioning on the availability of server 1 yields that the probability of 1 arrival during $[0, 2]$ at server 2 equals

$$\frac{3}{4}\mathbb{P}(N_2(2) = 1) + \frac{1}{4}\mathbb{P}(N(2) = 1) = 2e^{-8/3} + e^{-4}.$$

(c) [2 pt.] Let S_i be the service time of customer i, $i = A, B$. Then $\mathbb{P}(S_A < S_B) = \frac{\mu_1}{\mu_1+\mu_2}$. If A finishes its service, the remaining service time of B is still exponentially distributed with rate μ_2 due to the lack of memory. Hence,

$$\mathbb{P}(S_A + t < S_B) = \mathbb{P}(S_A < S_B)\mathbb{P}(S_B - S_A > t \mid S_B \geq S_A) = \frac{\mu_1}{\mu_1 + \mu_2}e^{-\mu_2 t}.$$

Exercise 4

(a) [3 pt.] Let $X(t)$ be the number of vehicles occupied at time t. Then, $\{X(t), t \geq 0\}$ is a continuous-time Markov chain on state space $\{0, 1, 2\}$. The state diagram with transition rates can be found in Figure 3.

The balance equations for states 0 and 2 are: $2p_0 = 4p_1$ and $8p_2 = 2p_1$. Expressing everything in terms of p_0 and using normalization gives $p_0(1 + 1/2 + 1/8) = 1$, such that $p_0 = 8/13$. Now, p_1 and p_2 follow directly given the required results.

(b) [2 pt.] The idle costs are $k(2p_0 + p_1) = \frac{20}{13}k$. The costs for both vehicles occupied are $sp_2 = \frac{1}{13}s$. Hence, the long-run average costs are $\frac{20}{13}k + \frac{1}{13}s$.

(c) [2 pt.] Let $Y(t)$ be the number of vehicles being cleaned at time t. Then $\{(X(t), Y(t)), t \geq 0\}$ is a CTMC with transition diagram given in Figure 4.

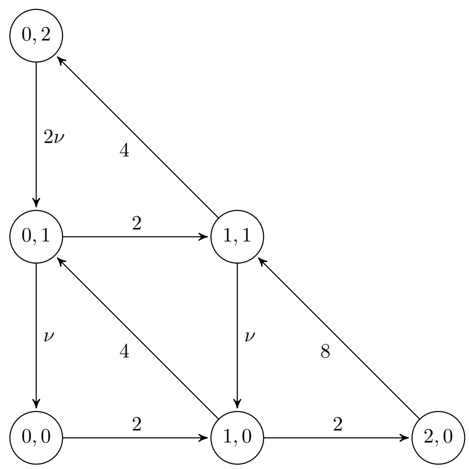

Figure 4: State diagram of the CTMC of exercise 4c.