Resit Stochastic Modeling (400646)

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This exam consists of five exercises. The use of books or a graphical calculator is not allowed. A formula sheet can be found at the end of the exam.

At each part it is indicated between square brackets how many points can be achieved for the corresponding part. The grade for this exam is given by $p/4 + 1$ where p is the total number of points earned by you. Please include your name and student number on all papers and motivate all your answers clearly.

Good luck!

Exercise 1

Consider a discrete-time Markov chain with state space $\{1, 2, 3, 4\}$ and matrix of transition probabilities

$$\begin{pmatrix} 1/3 & 1/3 & 1/3 & 0 \\ 2/3 & 1/3 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 2/3 & 1/3 \end{pmatrix}$$

(a) [2 pt.] Draw the state diagram of possible transitions and determine the classes of communicating states. What is $\mathbb{P}(X_2 = 1 \mid X_0 = 1)$?

(b) [3 pt.] Does the limiting distribution exist? And the occupancy distribution? If they exist, determine the limiting and/or the occupancy distribution.

(c) [2 pt.] Determine the expected number of transitions to reach state 3, starting in state 1.

Exercise 2

An electrical appliances shop carries in its range a certain type of washing machine. The demand for washing machines is stable over time. With probability $1/3$ there is no demand during a week, with probability $1/2$ the demand during a week is 1 and with probability $1/6$ the demand during a week is 2. The inventory control is as follows. When there is no inventory left at the end of the week there is an order of 3, whereas there is no order when washing machines are on stock. The time between placing and receiving the order can be neglected. The demand for washing machines when it is out of stock are lost.

(a) [2 p.] Formulate an appropriate discrete-time Markov chain to model the inventory level and specify the matrix of one-step transition probabilities.

(b) [2 p.] Give the balance equations. Also give the probability of lost sales during a week in terms of the limiting distribution.

Management decides to change the order policy. Orders are now placed at the end of every second week (i.e., orders are placed exactly once in every two weeks). When there are x items on stock when the order is placed, then the order size is $3 - x$. The time between placing and receiving the order can still be neglected.

(c) [3 pt.] Formulate an appropriate discrete-time Markov chain to model the inventory level of the new order policy and specify the matrix of one-step transition probabilities.

Exercise 3

Near a station, there are two competing supermarkets A and B. Customers arrive for a visit to either supermarket according to a Poisson process with rate λ. A customer goes to supermarket A with probability $2/3$ and to B with probability $1/3$, independent of other customers.

(a) [2 p.] What is the probability that no customers arrive to either supermarket during the next five time units? And what is the probability that during the next five time units at least 2 customers arrive for supermarket A, whereas no customer arrives for supermarket B?

(b) [2 p.] What is the probability that the next three customers go to supermarket A, and the first customer after that goes to supermarket B?

Suppose that the waiting time of both supermarkets can be well approximated by M/M/1 queues, with the arrival rates as given above. The service rate of supermarket A is 2 and the service rate of supermarket B is 1.

(c) [3 pt.] Determine the expected waiting time for both supermarkets in terms of λ. Which supermarket has a smaller expected waiting time? How can this difference be explained?

Exercise 4

Consider the M/M/s/3 queue. That is, customers arrive according to a Poisson process with rate λ. The service times are exponentially distributed with rate μ. There are s servers and 3 places in the system; customers finding all places occupied are rejected.

(a) [3 p.] Suppose that s is 3. Draw the state diagram with corresponding transition rates and determine the probability that a customer is rejected. Also, use Little's Law to determine the expected number of customers in the system (in terms of the probability that a customer is rejected).

(b) [3 p.] Suppose that s is 1. Draw the state diagram with corresponding transition rates and derive the probability that a customer is rejected.

(c) [2 p.] Suppose again that s is 1. Determine the probability that a customer is waiting at least t time units before he is taken into service.

Exercise 5

Consider an M/G/1 queue with arrival rate $1/4$. With probability $p \in [0, 1]$ the service time is exactly 2 and with probability $1 - p$ the service time follows an exponential distribution with rate $1/2$.

(a) [3 pt.] Show that the expected service time is 2 and the squared coefficient of variation of the service time is $1-p$. Also derive the expected waiting time $\mathbb{E}[W^q]$.

(b) [2 pt.] Make a sketch of $\mathbb{E}[W^q]$ as a function of $p \in [0, 1]$ and explain its behavior. Also determine the expected number of customers in the system.

When the queue becomes empty, the server is going on a break. The server is called back from its break when a new customer arrives to an empty queue. The time between calling the server back from its break and the moment that the server is actually ready to serve customers again follows an exponential random variable with rate η. Moreover, customers are served Last-Come First-Served, where customers in service are not preempted (the LCFS-NP discipline).

(c) [2 pt.] Give the arrival relation for the expected waiting time. Use this relation to derive the expected waiting time.

FORMULA SHEET

Erlang distribution Let S_n follow an Erlang(n, μ) distribution. The tail probability of S_n is then

$$\mathbb{P}(S_n > t) = \sum_{k=0}^{n-1} e^{-\mu t} \frac{(\mu t)^k}{k!}.$$

M/M/c queue The probability of waiting Π_W, expectation and distribution of the waiting time W^q and distribution of the sojourn time S

$$\begin{aligned}
\Pi_W &= \frac{(c\rho)^c/c!}{(1-\rho)\sum_{i=0}^{c-1}(c\rho)^i/i! + (c\rho)^c/c!} \\
\mathbb{E}(W^q) &= \Pi_W \frac{1}{c\mu(1-\rho)} \\
\mathbb{P}(W^q > t) &= \Pi_W e^{-c\mu(1-\rho)t} \\
\mathbb{P}(S > t) &= \frac{\Pi_W}{1-c(1-\rho)} e^{-c\mu(1-\rho)t} + \left(1 - \frac{\Pi_W}{1-c(1-\rho)}\right) e^{-\mu t}
\end{aligned}$$

M/M/c/c queue Blocking probability $B(c, a)$, with $a = \lambda/\mu = c\rho$, and relation between Erlang-B and Erlang-C:

$$B(c, a) = \frac{a^c/c!}{\sum_{i=0}^{c} a^i/i!} \qquad \text{and} \qquad \Pi_W = \frac{B(c, c\rho)}{1-\rho+\rho B(c, c\rho)}$$

M/G/1 queue Expected waiting time W^q for FCFS (Pollaczek-Khintchine)

$$\mathbb{E}(W^q) = \frac{\rho}{1-\rho} \frac{\mathbb{E}(B^2)}{2\mathbb{E}(B)} = \frac{1}{2}(1 + c_B^2)\mathbb{E}(B)\frac{\rho}{1-\rho}$$

Expected busy period

$$\mathbb{E}(BP) = \frac{\mathbb{E}(B)}{1-\rho}$$

G/M/1 queue Distribution number of customers found upon arrival π^* and expected waiting time W^q

$$\pi_j^* = (1-\sigma)\sigma^j \qquad \text{and} \qquad \mathbb{E}(W^q) = \frac{\sigma}{\mu(1-\sigma)}$$

with σ unique solution in $(0, 1)$ of $\sigma = \mathbb{E}\left[e^{-\mu(1-\sigma)A}\right]$ with A interarrival time

Residual life time Let X be the interarrival time and R be the residual life time. Distribution and expectation of the residual life time R

$$\mathbb{P}(R \leq x) = \frac{1}{\mathbb{E}(X)} \int_0^x \mathbb{P}(X > y)\mathrm{d}y \qquad \text{and} \qquad \mathbb{E}(R) = \frac{\mathbb{E}(X^2)}{2\mathbb{E}(X)}$$