Exam Stochastic Modeling (400646), period 2

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December 23, 2016, 8:45 - 11:30 hours

This exam consists of four exercises. The use of books or a graphical calculator is not allowed. A formula sheet can be found at the end of the exam.

At each part it is indicated between square brackets how many points can be achieved for the corresponding part. The grade for this exam is given by $p/3 + 1$ where p is the total number of points earned by you. Please include your name and student number on all papers and motivate all your answers clearly.

Good luck!

Exercise 1

Consider the following birth and death process with state space $\{0, 1, \ldots\}$. When the process is in state n, for $n = 0, 1, \ldots$, births occur with rate λ (representing arrivals). When the process is in state n, for $n = 1, 2, \ldots$, deaths occur with rate μ_n (representing the total service rate).

(a) [1 p.] Suppose that $\mu_n = \mu$ for $n = 2, 3, \ldots$ and $\mu_1 = p\mu$ for some $p \in (0, 1]$. For which values of λ, μ and p is the system stable?

(b) [4 pt.] Suppose again that $\mu_n = \mu$ for $n = 2, 3, \ldots$ and $\mu_1 = p\mu$ for some $p \in (0, 1]$. Specify the state diagram with the transition rates, and derive the limiting distribution of the number of customers in the system.

(c) [3 pt.] Suppose that $\mu_n = n\mu$ for $n = 1, 2, \ldots$. Specify the state diagram with the transition rates, and derive the limiting distribution of the number of customers in the system. Determine the probability that an arriving customer finds an empty system.

Exercise 2

Consider an M/G/1 queue with arrival rate 1. The service time B always consists of three consecutive steps. Step 1 takes an exponentially distributed time with rate μ. Step 2 always takes *exactly* $1/(2\mu)$ time units, and the time of step 3 follows an exponential distribution again with rate 2μ. We assume that $\mu > 2$.

(a) [3 pt.] Show that $\mathbb{E}B = 2/\mu$ and that the expected waiting time $\mathbb{E}[W^q]$ is given by

$$\mathbb{E}[W^q] = \frac{21}{16\mu}\frac{2}{\mu - 2}.$$

(b) [2 pt.] Make a sketch of $\mathbb{E}[W^q]$ as a function of μ ($\mu > 2$) and explain its behavior. Also determine the expected number of customers in the queue.

When the queue becomes empty, the server is going on a break. The server is called back from its break when a new customer arrives to an empty queue. The time between calling the server back from its break and the moment that the server is actually ready to serve customers again follows an exponential random variable with rate η.

(c) [2 pt.] Give the arrival relation, relating the expected waiting time and expected number of customers in the queue. Derive the new expected waiting time from this relation.

Exercise 3

A single data-science specialist performs highly specialized consultancy tasks for a group of N customers. The time to complete a task follows an exponential distribution with rate α. For each customer it holds that the time until a new task arises (from the moment that the previous task has been completed) is exponentially distributed with rate β. When the consultant has multiple tasks, then he handles them on a first-come first-served (FCFS) basis. Customers do not call for new tasks when they have uncompleted tasks at the consultant.

(a) [2 pt.] Formulate a continuous-time Markov chain to analyze the situation and specify the state diagram with the transition rates.

(b) [2 pt.] To which queueing model does the above Markov chain correspond? Determine the probability that a customer has to wait due to the consultant working on other tasks first.

The time to complete a task turns out to be slightly different. With probability $1/2$ a task takes an exponentially distributed time with rate γ_1 and with probability $1/2$ a task takes an exponentially distributed time with rate γ_2, with $\gamma_1 \neq \gamma_2$.

(c) [2 pt.] At an arbitrary moment when the consultant is busy, the manager wants to know the remaining time R until the consultant completes the task he is currently working on. Determine the probability that this time R is at least t time units.

(d) [2 pt.] Suppose that $N = 2$. Formulate a continuous-time Markov chain for the situation with the new completion times of tasks. Specify the state diagram with the transition rates.

Exercise 4

Consider a queueing system with two identical servers. The arrivals of customers are scheduled such that the interarrival times are *exactly* 1 time unit. Arriving customers are assigned to the two servers in an alternating fashion (i.e., numbering the arrivals consecutively, all even-numbered customers go to server 1, and odd-numbered customers go to server 2). The service times are exponentially distributed with rate μ. Customers are not allowed to change queue during waiting. For ease, we focus on the number of customers in front of server 1.

(a) [3 pt.] For which value of μ is the system stable? Give the limiting distribution of the number of customers in front of server 1 at the moment that a customer arrives at that queue.

(b) [1 pt.] Suppose that there are 3 customers in front of the server when a new customer arrives. Give an expression for the probability that this customer has to wait at least t time units.

FORMULA SHEET

Erlang distribution Let S_n follow an Erlang(n, μ) distribution. The tail probability of S_n is then

$$\mathbb{P}(S_n > t) = \sum_{k=0}^{n-1} e^{-\mu t} \frac{(\mu t)^k}{k!}.$$

M/M/c queue The probability of waiting Π_W, expectation and distribution of the waiting time W^q and distribution of the sojourn time S

$$\begin{aligned}
\Pi_W &= \frac{(c\rho)^c/c!}{(1-\rho)\sum_{i=0}^{c-1}(c\rho)^i/i! + (c\rho)^c/c!} \\
\mathbb{E}(W^q) &= \Pi_W \frac{1}{c\mu(1-\rho)} \\
\mathbb{P}(W^q > t) &= \Pi_W e^{-c\mu(1-\rho)t} \\
\mathbb{P}(S > t) &= \frac{\Pi_W}{1-c(1-\rho)} e^{-c\mu(1-\rho)t} + \left(1 - \frac{\Pi_W}{1-c(1-\rho)}\right) e^{-\mu t}
\end{aligned}$$

M/M/c/c queue Blocking probability $B(c,a)$, with $a = \lambda/\mu = c\rho$, and relation between Erlang-B and Erlang-C:

$$B(c,a) = \frac{a^c/c!}{\sum_{i=0}^{c} a^i/i!} \qquad \text{and} \qquad \Pi_W = \frac{B(c,c\rho)}{1-\rho+\rho B(c,c\rho)}$$

M/G/1 queue Expected waiting time W^q for FCFS (Pollaczek-Khintchine)

$$\mathbb{E}(W^q) = \frac{\rho}{1-\rho}\frac{\mathbb{E}(B^2)}{2\mathbb{E}(B)} = \frac{1}{2}(1+c_B^2)\mathbb{E}(B)\frac{\rho}{1-\rho}$$

Expected busy period

$$\mathbb{E}(BP) = \frac{\mathbb{E}(B)}{1-\rho}$$

G/M/1 queue Distribution number of customers found upon arrival π^* and expected waiting time W^q

$$\pi_j^* = (1-\sigma)\sigma^j \qquad \text{and} \qquad \mathbb{E}(W^q) = \frac{\sigma}{\mu(1-\sigma)}$$

with σ unique solution in $(0,1)$ of $\sigma = \mathbb{E}\left[e^{-\mu(1-\sigma)A}\right]$ with A interarrival time

Residual life time Let X be the interarrival time and R be the residual life time. Distribution and expectation of the residual life time R

$$\mathbb{P}(R \le x) = \frac{1}{\mathbb{E}(X)}\int_0^x \mathbb{P}(X > y)\mathrm{d}y \qquad \text{and} \qquad \mathbb{E}(R) = \frac{\mathbb{E}(X^2)}{2\mathbb{E}(X)}$$