Exam Stochastic Modeling (400646), period 2 - Solutions

The solutions are always provisionary

Vrije Universiteit Amsterdam
Faculty of Sciences, Department of Mathematics

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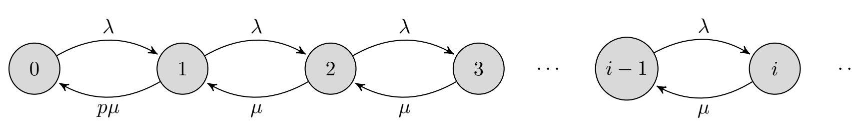
Exercise 1.

(a) The system is stable for $\lambda/\mu < 1$.

(b) The state diagram with the transition rates is as follows:



Figuur 1: State diagram Exercise 1(b).

The balance equations are then as follows:

$$\begin{aligned} \lambda p_0 &= p\mu p_1 \\ \lambda p_{i-1} &= \mu p_i \qquad i = 2, 3, \ldots \end{aligned}$$

Expressing in terms of p_0 yields, for $i = 1, 2, \ldots$,

$$p_i = \frac{\lambda}{\mu} p_{i-1} = \left(\frac{\lambda}{\mu}\right)^{i-1} p_1 = \frac{1}{p}\left(\frac{\lambda}{\mu}\right)^{i} p_0.$$

Normalization provides

$$p_0 + p_0 \sum_{i=1}^{\infty} \frac{1}{p}\left(\frac{\lambda}{\mu}\right)^{i} = 1.$$

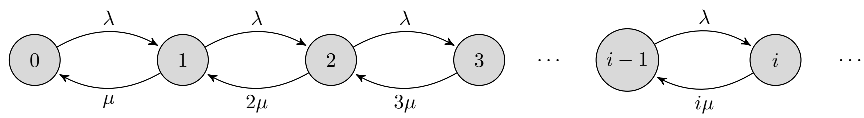
Working out the summation yields the required p_0:

$$p_0 = \frac{1 - {}^{\lambda}/_{\mu}}{1 - {}^{\lambda}/_{\mu}(1 - {}^{1}/_{p})}.$$

(c) The state diagram with the transition rates is presented in Figure 2.

You may recognize this directly as an M/M/∞ queue. The analysis then proceeds as follows. The balance equations are $\lambda p_{i-1} = i\mu p_i$, thus $p_i = \frac{\lambda}{i\mu} p_{i-1} = \ldots = \frac{({}^{\lambda}/_{\mu})^i}{i!} p_0$. Normalization gives

$$p_0 = \left[\sum_{i=0}^{\infty} \frac{({}^{\lambda}/_{\mu})^i}{i!}\right]^{-1} = e^{-\lambda/\mu},$$



Figuur 2: State diagram Exercise 1(c).

such that p_i has a Poisson distribution with rate λ/μ.

Finally, due to PASTA, the probability that an arriving customer finds an empty system is $\pi_0 = p_0 = e^{-\lambda/\mu}$.

Exercise 2.

(a) The expectation and variance of the service time B can be calculated as

$$\begin{aligned} \mathbb{E}B &= \frac{1}{\mu} + \frac{1}{2\mu} + \frac{1}{2\mu} = \frac{2}{\mu} \\ \mathbb{V}\text{ar}B &= \frac{1}{\mu^2} + 0 + \frac{1}{(2\mu)^2} = \frac{5}{4\mu^2}. \end{aligned}$$

Thus $c_B^2 = \frac{\frac{5}{4\mu^2}}{\frac{2^2}{\mu^2}} = \frac{5}{16}$ and the load $\rho = \lambda \mathbb{E}B = \frac{2}{\mu}$. Now, using Pollaczek-Khinchine

$$\mathbb{E}W^q = \frac{1}{2}(1 + c_B^2)\mathbb{E}B\frac{\rho}{1-\rho} = \frac{1}{2}\left(1 + \frac{5}{16}\right)\frac{2}{\mu}\frac{2/\mu}{1 - 2/\mu}.$$

Some rewriting gives the desired result.

(b) Make a sketch. Note that μ only affects the mean service time and thus also the load, and not the variability in the service duration. Now observe that (i) if $\mu \downarrow 2$, the system tends to the boundary of the stability region and $\mathbb{E}W^q$ explodes, and (ii) the system load decreases as μ increasas (and c_B^2 is independent of μ) and thus $\mathbb{E}W^q$ decreases in μ.

Finally, using Little's law gives

$$\mathbb{E}L^q = \lambda \mathbb{E}W^q = \frac{21}{16\mu}\frac{2}{\mu - 2}.$$

(c) The arrival relation is (using PASTA)

$$\mathbb{E}W^q = \mathbb{E}L^q \times \mathbb{E}B + \rho \mathbb{E}R + (1-\rho)\frac{1}{\eta},$$

with $\mathbb{E}R = \frac{1}{2}(1 + c_B^2)\mathbb{E}B = \frac{21}{16\mu}$ the expected residual service time (given that it is positive). Using Little's law $\mathbb{E}L^q = \lambda \mathbb{E}W^q = \mathbb{E}W^q$, we obtain

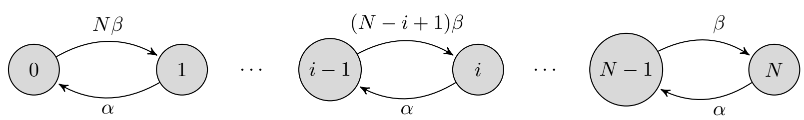
$$\mathbb{E}W^q = \rho \mathbb{E}W^q + \rho \mathbb{E}R + (1-\rho)\frac{1}{\eta}.$$

Solving for $\mathbb{E}W^q$ and using the expressions for ρ and $\mathbb{E}R$ yields

$$\lambda \mathbb{E}W^q = \frac{21}{16\mu}\frac{2}{\mu - 2} + \frac{1}{\eta}.$$

Exercise 3.

(a) Define $X(t)$ as the number of uncompleted tasks at the consultant at time t. Then, $\{X(t), t \geq 0\}$ is a CTMC on $I = \{0, 1, \ldots, N\}$ with transition diagram as presented in Figure 3.



Figuur 3: State diagram of Exercise 3(a).

(b) By inspecting the state diagram you see that $N - X(t)$ corresponds to an M/M/N/N model (Erlang B); alternatively, you may define $Y(t)$ = number of satisfied customers at time t, in which case it directly is an Erlang B model.

Now, the probability that a customer has to wait is $1 - p_0$ which is $1 - p_{block}$ in the Erlang B model with arrival rate α and service rate β; thus, the required probability is (using the formula sheet), with $a = \alpha/\beta$

$$1 - \frac{(a)^N/N!}{\sum_{i=0}^{N}(a)^i/i!}.$$

(c) Note that we in fact have a hyperexponential service time. The key is now to condition on the type of exponential. First, the mean time to complete a task is

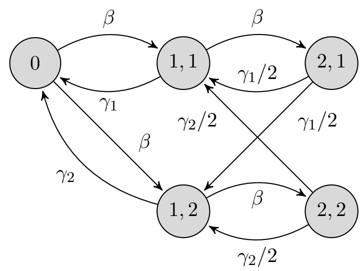
$$\mathbb{E}X = \frac{1}{2\gamma_1} + \frac{1}{2\gamma_2}.$$

Now, the distribution of the remaining time R is

$$\begin{aligned}
\mathbb{P}(R \leq t) &= \frac{1}{\mathbb{E}X}\int_{y=0}^{t} \mathbb{P}(X > y)\mathrm{d}y \\
&= \frac{1}{\mathbb{E}X}\left[\frac{1}{2}\int_{y=0}^{t} e^{-\gamma_1 y}\mathrm{d}y + \frac{1}{2}\int_{y=0}^{t} e^{-\gamma_2 y}\mathrm{d}y\right] \\
&= \frac{1}{\mathbb{E}X}\left[\frac{1}{2\gamma_1}(1 - e^{-\gamma_1 t}) + \frac{1}{2\gamma_2}(1 - e^{-\gamma_2 t})\right].
\end{aligned}$$

As $\mathbb{E}X$ is calculated above, this completes the analysis. It is possible to rewrite this probability, e.g. as in Exercise 53 of the tutorials.

(d) To maintain the Markov property, we also need to keep track of the type of task that is in service. For instance, define $Z(t)$ = the type of task in service at time t. Then $\{(X(t), Z(t)), t \geq 0\}$ is a CTMC; the transition diagram is given in Figure 4. Note that when a new service starts (due to an arrival to an empty system or a service completion with $X(t) = 2$), it is determined which type of task is taken into service.



Figuur 4: State diagram of Exercise 3(d).

Exercise 4.

(a) Note that the interarrival times of customers to queue 1 are *exactly* equal to 2 (or you may say that the long-run average arrival rate is 0.5). Hence, the system is stable for $\frac{1}{2}\frac{1}{\mu} < 1$ or, equivalently, $\mu > 1/2$.

Observe that queue 1 behaves as a D/M/1 queue with interarrival time 2 and service rate μ. Thus the limiting distribution of the number of customers in front of server 1 is

$$\pi_j^* = (1-\sigma)\sigma^j,$$

where σ is the unique solution in $(0,1)$ of the equation

$$\sigma = e^{-\mu(1-\sigma)2}.$$

(b) The waiting time is then a sum of three exponential distributions and thus follows an Erlang(3,μ) distribution. The probability that the waiting time exceeds t is then

$$\sum_{k=0}^{2} e^{-\mu t}\frac{(\mu t)^k}{k!} = e^{-\mu t}(1 + \mu t + \frac{1}{2}(\mu t)^2).$$