Midterm Stochastic Modeling (400646)

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October 24, 2016, 12:00 - 14:00 hours

This (midterm) exam consist of four exercises. The use of books or a graphical calculator is not allowed.

At each part it is indicated between square brackets how many points can be achieved for the corresponding part. The grade for the midterm is given by $p/3 + 1$ where p is the total number of points earned by you. Please include your name and student number on all papers and motivate all your answers clearly.

Good luck!

Exercise 1

An international company has an internal consultant, which is specialized in big data solutions. The company has branches in three countries: G (Germany), N (Netherlands) and U (USA). The consultant remains at one branch for exactly one week, and then moves to another branch. After a stay in G, the consultant goes to N with probability $2/5$. After a stay in N, the consultant goes to G with probability $1/2$. After a stay in U, the consultant always goes to G. From a cost perspective, the company is also interested in traveling expenses. Assume that flights overseas (from U to G or N, and vice versa) costs 1000, whereas European travel (between G and N) costs 200; these are prices for one-way tickets.

(a) [3 pt.] Formulate an appropriate Markov chain to analyze the location of the internal consultant. What is the long-run fraction of time that the consultant is at a location in Europe?

(b) [2 pt.] Determine the long-run average traveling costs per week.

Due to the amount of work at the branches G and U, the company decided that the consultant will sometimes stay at G for two consecutive weeks (i.e., the consultant remains at the same location for either 1 or 2 weeks); the same holds for U. With probability p_i, $i \in \{G, U\}$, the consultant stays for a second week at location i (after which the consultant moves to another branch according to the transition probabilities mentioned above).

(c) [2 pt.] Formulate an appropriate Markov chain and specify the matrix of one-step transition probabilities.

Exercise 2

Consider a discrete-time Markov chain with state space $\{1, 2, 3, 4, 5\}$ and matrix of transition probabilities

$$\begin{pmatrix} 0 & 1/3 & 2/3 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) [3 pt.] Draw the state diagram of possible transitions and determine $\mathbb{P}(X_n = 1 \mid X_0 = 1)$ for $n = 1, 2, 3, 4$.

(b) [3 pt.] Does the limiting distribution exist? And the occupancy distribution? If they exist, determine the limiting and/or the occupancy distribution.

(c) [2 pt.] Determine the expected number of transitions to reach state 3, starting in state 1.

Exercise 3

A fashion outlet sells the last few sweaters of the season in sizes M and L. On stock is 1 sweater of size M and 2 sweaters of size L (and sweaters are no longer replenished). Customers arrive to the outlet according to a Poisson process with rate λ per day; of those customers 10% is interested in the specific sweater and would like to buy it. Of the interested customers, 25% needs size M and 75% needs size L.

(a) [2 pt.] What is the probability that during one day no customers arrive that are interested in the sweater? And what is the joint probability that the first half of the day no customers arrive that are interested in buying the sweater and during the second half of the day at least two customers arrive that are interested?

(b) [2 pt.] What is the probability that size L is out of stock before size M is out of stock?

Suppose that customers needing a sweater of size M buy a sweater of size L when size M is out of stock. Customers needing a sweater of size L do *not* buy a sweater of size M when size L is out of stock (as the sweater is too small).

(c) [3 pt.] What is the probability that sweaters of size M are left at the end of the day? Also, determine the probability that sweaters of size L are left at the end of the day, by conditioning on the number of customers needing a sweater of size M.

Exercise 4

A small museum has two rooms that customers may visit. The admission policy is such that there are always exactly 3 customers in total in the museum. Customers first visit room 1 and stay there for an exponential time with mean $1/\mu_1$. From room 1, they go to room 2 and stay there for an exponential time with mean $1/\mu_2$. When customers leave room 2, they leave the museum, and a new customer is directly admitted to room 1 (from the excessively long waiting line in front of the museum) such that the total number of customers remains 3. All sojourn times in the rooms are assumed to be independent. The museum is interested in the distribution of the number of customers over the two rooms.

(a) [2 pt.] Suppose that all customers are in room 1. Show that the probability that room 2 stays empty for t more time units is equal to $e^{-3\mu_1 t}$.

(b) [3 pt.] Formulate an appropriate continuous-time Markov chain to analyze the distribution of the number of customers at the museum. Specify the state diagram and transition rates. Also, give the balance equations.