Midterm Stochastic Modeling (400646) - Solutions

The solutions are always provisionary

Vrije Universiteit Amsterdam
Faculty of Sciences, Department of Mathematics

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Exercise 1

(a) [3 pt.] Let X_n be the location of the consultant in week n. Then $\{X_n, n = 0, 1, \ldots\}$ is a discrete-time Markov chain (DTMC) on the state space $I = \{G, N, U\}$. The state diagram with transition probabilities is given in Figure 1.

To determine the long-run fraction of time the consultant is in Europe, we derive the occupancy distribution, which starts with the balance equations (for states U and N):

$$\pi_U = \frac{3}{5}\pi_G + \frac{1}{2}\pi_N$$
$$\pi_N = \frac{2}{5}\pi_G$$

Substituting the second in the first equation yields $\pi_U = (\frac{3}{5} + \frac{1}{2}\frac{2}{5})\pi_G = \frac{4}{5}\pi_G$. Using normalization gives $\pi_G(1+\frac{4}{5}+\frac{2}{5}) = 1$, hence $\pi_G = \frac{5}{11}$ (and thus $\pi_N = \frac{2}{11}$ and $\pi_U = \frac{4}{11}$). The long-run fraction of time the consultant is in Europe is then $\pi_G + \pi_N = {}^7/_{11}$

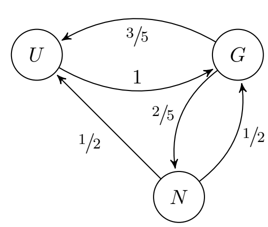

Figure 1: State diagram of the DTMC of exercise 1a.

(b) [2 pt.] The long-run average cost per week is

$$1000 \times \left(\pi_U \times 1 + \pi_G \times \frac{3}{5} + \pi_N \times \frac{1}{2}\right) + 200 \times \left(\pi_N \times \frac{1}{2} + \pi_G \times \frac{2}{5}\right)$$
$$= 1000 \times \frac{8}{11} + 200 \times \frac{3}{11} = \frac{8600}{11}$$

The second equality follows from the occupancy distribution determined in (a).

(c) [2 pt.] Define Y_n as the number of consecutive weeks that the consultant is at the same location in week n. Then $\{(X_n, Y_n), n = 0, 1, \ldots\}$ is a DTMC on state space $\{(G,1),(G,2),(N,1),(U,1),(U,2)\}$. The one-step transition probability matrix reads

$$\mathbf{P} = \begin{pmatrix} 0 & p_G & (1-p_G)^2/5 & (1-p_G)^3/5 & 0 \\ 0 & 0 & ^2/_5 & ^3/_5 & 0 \\ ^1/_2 & 0 & 0 & ^1/_2 & 0 \\ (1-p_U) & 0 & 0 & 0 & p_U \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Exercise 2

(a) [3 pt.] See Figure 2 for the state diagram. The transient probabilities are

$$\mathbb{P}(X_1 = 1 \mid X_0 = 1) = 0$$
$$\mathbb{P}(X_2 = 1 \mid X_0 = 1) = \frac{1}{3} + \frac{2}{3}\frac{1}{2} = \frac{2}{3}$$
$$\mathbb{P}(X_3 = 1 \mid X_0 = 1) = 0$$
$$\mathbb{P}(X_4 = 1 \mid X_0 = 1) = \left(\frac{2}{3}\right)^2 + 1 \times \frac{2}{3}\frac{1}{2} = \frac{4}{9} + \frac{1}{3} = \frac{7}{9}$$

(b) [3 pt.] The DTMC is periodic, so the limiting distribution does not exist. The DTMC is irreducible and the state space is finite, so the occupancy distribution does exist.

To determine the latter, we solve the set of balance equations

$$\pi_1 = \pi_2 + \frac{1}{2}\pi_3 + \pi_5, \quad \pi_2 = \frac{1}{3}\pi_1,$$
$$\pi_3 = \frac{2}{3}\pi_1, \quad \pi_4 = \frac{1}{2}\pi_3 \ \left(= \frac{1}{3}\pi_1\right), \quad \pi_5 = \pi_4 \ \left(= \frac{1}{3}\pi_1\right)$$

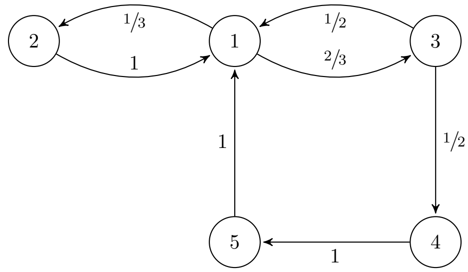

Figure 2: State diagram of the DTMC of exercise 2a.

Normalization provides $\pi_1(1 + \frac{1}{3} + \frac{2}{3} + \frac{1}{3} + \frac{1}{3}) = 1$, hence $\pi_1 = \frac{3}{8}$. So, the occupancy distribution is $\hat{\pi} = (3/8, 1/8, 1/4, 1/8, 1/8)$.

(c) [2 pt.] Let m_i be the expected number of transitions to reach state 3, given that the current state is i. We are looking for m_1. This follows from the equations:

$$m_1 = 1 + \frac{1}{3} m_2$$
$$m_2 = 1 + m_1$$

Hence, the final answer is $m_1 = 2$.

Exercise 3

Define $N_T(t)$, $N_M(t)$, and $N_L(t)$ as the arriving number of interested customers during $[0, t)$ in total (with t in days), needing size M, and needing size L, respectively. Due to thinning, it holds that $N_T(t)$ PP($\lambda/10$), $N_M(t)$ PP($\lambda/40$), and $N_L(t)$ PP($3\lambda/40$).

(a) [2 pt.] From the above, the probability that no interested customer arrives is

$$\mathbb{P}(N_T(t) = 0) = e^{-\lambda/10}.$$

Due to stationary and independent increments, the second (joint) probability is (where $N_T(s, t)$ are the number of type T arrivals during $[s, t)$)

$$\begin{aligned}\mathbb{P}(N_T(0.5) = 0; N_T(0.5, 1) \geq 2) &= \mathbb{P}(N_T(0.5) = 0)\mathbb{P}(N_T(0.5) \geq 2) \\ &= e^{-\lambda/20}\left(1 - e^{-\lambda/20}(1 + \lambda/20)\right)\end{aligned}$$

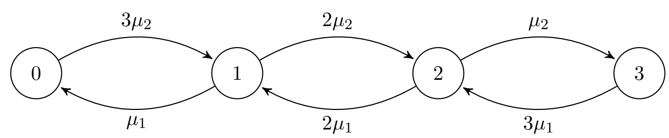

Figure 3: State diagram of the CTMC of exercise 4b.

(b) [2 pt.] Note that the times between sales are exponential. Size L is out of stock before size M in case the time between sales of L 'beats' those of size M twice, i.e.

$$\left(\frac{3\lambda/40}{\lambda/40 + 3\lambda/40}\right)^2 = \left(\frac{3}{4}\right)^2 = \frac{9}{16}$$

(c) [3 pt.] The probability that sweater of size M are left at the end of the day is

$$\mathbb{P}(N_M(1) = 0) = e^{-\lambda/40}$$

For the probability that sweaters of size L are left at the end of the day, we may write

$$\begin{aligned}
&\mathbb{P}(N_M(1) \leq 1, N_L(1) \leq 1) + \mathbb{P}(N_M(1) = 2, N_L(1) = 0) \\
&\quad = e^{-\lambda/40}(1 + \lambda/40) \times e^{-3\lambda/40}(1 + 3\lambda/40) + \mathbb{P}(N_M(1) = 2)\mathbb{P}(N_L(1) = 0) \\
&\quad = e^{-\lambda/10}\left[(1 + \lambda/40)(1 + 3\lambda/40) + \frac{(\lambda/40)^2}{2}\right]
\end{aligned}$$

Exercise 4

(a) [2 pt.] Let X_i be the (remaining) time that customer i remains in room 1, $i = 1, 2, 3$. Note that $X_i \sim \text{Exp}(\mu_1)$. Then

$$\mathbb{P}(\text{room 2 empty during } [0,t)) = \mathbb{P}(X_1 > t, X_2 > t, X_3 > t) = \left(e^{-\mu_1 t}\right)^3 = e^{-3\mu_1 t}$$

(b) [3 pt.] Let $X(t)$ be the number of customers in room 1 at time t. Then, $\{X(t), t \geq 0\}$ is a continuous-time Markov chain on state space $\{0, 1, 2, 3\}$. The state diagram with transition rates can be found in Figure 3.

The balance equations are

$$\begin{aligned}
3\mu_2 p_0 &= \mu_1 p_1 \\
(\mu_1 + 2\mu_2)p_1 &= 3\mu_2 p_0 + 2\mu_1 p_2 \\
(2\mu_1 + \mu_2)p_2 &= 2\mu_2 p_1 + 3\mu_1 p_3 \\
3\mu_1 p_3 &= \mu_2 p_2
\end{aligned}$$