Exam Stochastic Modeling (400646), period 2

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December 16, 2015, 15:15 - 17:15 hours

This (midterm) exam consist of four exercises. The use of books or a graphical calculator is not allowed. A formula sheet can be found at the end of the exam.

At each part it is indicated between square brackets how many points can be achieved for the corresponding part. The grade for this exam is given by $p/3 + 1$ where p is the total number of points earned by you. Please include your name and student number on all papers and motivate all your answers clearly.

Good luck!

Exercise 1

Consider the following queueing system. Potential customers arrive according to a Poisson process with rate λ at a queueing system with a single server. When there are customers present upon arrival, a newly arriving customer joins the queue with probability p (and thus leaves immediately with probability $1-p$). Assume that the service times are exponentially distributed with rate μ.

(a) [1 p.] For which values of λ, μ and p is the system stable?

(b) [4 pt.] Let p_j, $j \in \{0, 1, \ldots\}$, be the probability of j customers in the system in equilibrium. Specify the state diagram with the transition rates. Express p_j, for $j = 1, 2 \ldots$, in terms of p_0 and show that

$$p_0 = \left[1 + \frac{\frac{\lambda}{\mu}}{1 - \frac{\lambda p}{\mu}}\right]^{-1}.$$

(c) [2 pt.] Suppose that $p = 0$. Determine the limiting distribution of the number of customers in the system and determine the fraction of customers that are lost.

Exercise 2

The Fire Department in the region of Amsterdam wants to obtain insight in the demand for fire-fighting vehicles. For now, one assumes a practically unlimited capacity. Assume that fire alarms, for which a fire-fighting vehicle is required, occur according to a Poisson process with a rate of 3/2 per hour. For each alarm, exactly one vehicle is required. The 'integral time' that a fire-fighting vehicle is busy with an alarm comprises of turn out, save & rescue, and return to fire station.

(a) [3 pt.] Suppose that the 'integral time' follows an exponential distribution with an average of 3 hours. Formulate an appropriate continuous-time Markov chain for the number of busy fire-fighting vehicles and specify the state diagram with the transition rates.

(b) [2 pt.] Suppose that the time for turn out, the time for save & rescue, and the time for returning to the fire station each follow an exponential distribution with an average of 1 hour. What is then the distribution of the 'integral time'? How can the process of occupied fire-fighting vehicles now be transformed into a continuous-time Markov chain (give a state description and the corresponding state space)?

One decides to deploy c fire-fighting vehicles. When a fire alarm occurs while all c fire-fighting vehicles are occupied, then a neighboring region takes over the alarm (and the alarm is 'lost' for the Amsterdam region).

(c) [3 pt.] With which known queueing model does this system correspond when the 'integral times' are exponentially distributed with an average of 3 hours? Give an expression for the fraction of fire alarms that is taken over by neighboring regions. What is the impact on the fraction of fire alarms taken over when the times are distributed as in part (b)?

Exercise 3

Consider an M/G/1 queue with arrival rate $3/(4b)$. The service time B is exactly equal to $b > 0$.

(a) [2 pt.] Show that the expected waiting time $\mathbb{E}[W^q]$ is given by

$$\mathbb{E}[W^q] = \frac{3}{2}b.$$

(b) [2 pt.] Verify that for the expected number of customers in the queue it holds that $\mathbb{E}[L^q] = \frac{9}{8}$. Make a sketch of $\mathbb{E}[W^q]$ and $\mathbb{E}[L^q]$ as a function of b and explain its behavior.

There appears to be some variation in service times after all. Suppose that the service time is equal to $\frac{1}{2}b$ with probability $\frac{1}{2}$ and equal to $\frac{3}{2}b$ also with probability $\frac{1}{2}$.

(c) [2 pt.] Determine the expected waiting time and the expected sojourn time.

Exercise 4

Consider an M/M/2 queueing system: customers arrive according to a Poisson process with rate λ. The service times follow an exponential distribution with rate μ. There are 2 servers and the size of the waiting room is unlimited. We assume that $\lambda < 2\mu$ and customers are served according to the FCFS discipline.

(a) [1 pt.] A new customer arrives at an arbitrary instant. Let R be the time until the next service completion, given that both servers are occupied. Determine the probability that the next t time units there is no service completion, or in other words $\mathbb{P}(R > t)$.

(b) [3 pt.] Argue that the expected waiting time $\mathbb{E}[W^q]$ and the expected number of customers in the queue $\mathbb{E}[L^q]$ are related via

$$\mathbb{E}[W^q] = \mathbb{E}[L^q]/(2\mu) + \Pi_W/(2\mu),$$

where Π_W is the probability of waiting. Use the above to determine the expected waiting time in terms of Π_W.

One decides to modify the service discipline into non-preemptive LCFS (customers in service are thus not interrupted).

(c) [2 pt.] Argue that the arrival relation is now given by

$$\mathbb{E}[W^q] = \Pi_W \left(\frac{1}{2\mu} + \frac{1}{2\mu} \lambda \mathbb{E}[BP] \right),$$

where $\mathbb{E}[BP]$ denotes the expected busy period during which both servers are working without interruption.

FORMULA SHEET[1]

Erlang distribution Let S_n follow an Erlang(n, μ) distribution. The tail probability of S_n is then

$$\mathbb{P}(S_n > t) = \sum_{k=0}^{n-1} e^{-\mu t} \frac{(\mu t)^k}{k!}.$$

M/M/c queue The probability of waiting Π_W, expectation and distribution of the waiting time W^q and distribution of the sojourn time S

$$\begin{aligned}
\Pi_W &= \frac{(c\rho)^c/c!}{(1-\rho)\sum_{i=0}^{c-1}(c\rho)^i/i! + (c\rho)^c/c!} \\
\mathbb{E}(W^q) &= \Pi_W \frac{1}{c\mu(1-\rho)} \\
\mathbb{P}(W^q > t) &= \Pi_W e^{-c\mu(1-\rho)t} \\
\mathbb{P}(S > t) &= \frac{\Pi_W}{1-c(1-\rho)} e^{-c\mu(1-\rho)t} + \left(1 - \frac{\Pi_W}{1-c(1-\rho)}\right) e^{-\mu t}
\end{aligned}$$

M/M/c/c queue Blocking probability $B(c,a)$, with $a = \lambda/\mu = c\rho$, and relation between Erlang-B and Erlang-C:

$$B(c,a) = \frac{a^c/c!}{\sum_{i=0}^{c} a^i/i!} \qquad \text{and} \qquad \Pi_W = \frac{B(c,c\rho)}{1-\rho+\rho B(c,c\rho)}$$

M/G/1 queue Expected waiting time W^q for FCFS (Pollaczek-Khintchine)

$$\mathbb{E}(W^q) = \frac{\rho}{1-\rho}\frac{\mathbb{E}(B^2)}{2\mathbb{E}(B)} = \frac{1}{2}(1+c_B^2)\mathbb{E}(B)\frac{\rho}{1-\rho}$$

Expected busy period

$$\mathbb{E}(BP) = \frac{\mathbb{E}(B)}{1-\rho}$$

G/M/1 queue Distribution number of customers found upon arrival π^* and expected waiting time W^q

$$\pi_j^* = (1-\sigma)\sigma^j \qquad \text{and} \qquad \mathbb{E}(W^q) = \frac{1}{\mu(1-\sigma)}$$

with σ unique solution in $(0,1)$ of $\sigma = \mathbb{E}\left[e^{-\mu(1-\sigma)A}\right]$ with A interarrival time

Residual life time Let X be the interarrival time and R be the residual life time. Distribution and expectation of the residual life time R

$$\mathbb{P}(R \leq x) = \frac{1}{\mathbb{E}(X)} \int_0^x \mathbb{P}(X > y)\mathrm{d}y \qquad \text{and} \qquad \mathbb{E}(R) = \frac{\mathbb{E}(X^2)}{2\mathbb{E}(X)}$$

[1] This is the formula sheet as you will have at the exam of December 2016