Exam Stochastic Modeling (400646) - Solutions

The solutions are always provisionary

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Exercise 1.

(a) The system is stable for $\lambda p/\mu < 1$.

(b) The state diagram with the transition rates is as follows:

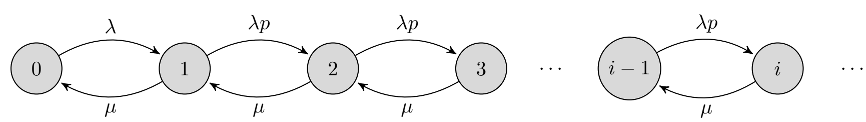

Figure 1: State diagram Exercise 1(b).

The balance equations are then as follows:

$$\begin{aligned} \lambda p_0 &= \mu p_1 \\ \lambda p \times p_{i-1} &= \mu p_i \qquad i = 2, 3, \ldots \end{aligned}$$

Expressing in terms of p_0 yields, for $i = 1, 2, \ldots$,

$$p_i = \frac{\lambda p}{\mu} p_{i-1} = \left(\frac{\lambda p}{\mu}\right)^{i-1} p_1 = \left(\frac{\lambda p}{\mu}\right)^{i} \frac{1}{p} p_0.$$

Normalization provides

$$p_0 + p_0 \sum_{i=1}^{\infty} \left(\frac{\lambda p}{\mu}\right)^{i} \frac{1}{p} = 1.$$

Working out the summation yields the required p_0.

(c) For $p = 0$ it is typically the easiest to draw the state diagram again and set up the balance equations from there: $\lambda p_0 = \mu p_1$. This directly gives $p_1 = \lambda p_0/\mu$. Applying normalization yields $(1 + \lambda/\mu)p_0 = 1$, or

$$p_0 = \frac{\mu}{\lambda + \mu}, \quad \text{and} \quad p_1 = \frac{\lambda}{\lambda + \mu}.$$

Due to PASTA it holds that the fraction of customers lost is $p_1 = \lambda/(\lambda + \mu)$.

Exercise 2.

(a) Define $X(t)$ as the number of busy fire-fighting vehicles at time t. Then, $\{X(t), t \geq 0\}$ is a CTMC on $I = \{0, 1, \dots\}$ with transition diagram as presented in Figure 2.

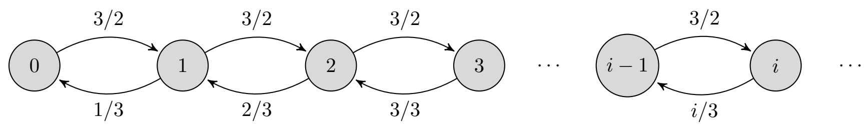

Figure 2: State diagram of Exercise 2(a).

(b) The integral time is then the sum of 3 exponential random variables with rate 1, or in other words it follows an Erlang(3,1) distribution.

There are multiple options to model this as a CTMC. The simplest is
$Y_i(t)$ = the number if fire-fighting vehicles in phase i, $i = 1, 2, 3$.
Then $\{(Y_1(t), Y_2(t), Y_3(t)), t \geq 0\}$ is a CTMC on $I = \{0, 1, \dots\}^3$.

(c) This corresponds with an M/M/c/c model, or Erlang B.
The offered load (average amount of work offered) is here $a = \lambda \mathbb{E}B = 3/2 \times 3 = 4.5$.
The fraction of alarms taken over by neighboring regions is then

$$\frac{(4.5)^c/c!}{\sum_{j=0}^{c}(4.5)^j/j!}.$$

The Erlang B model is insensitive to the distribution of the service time (with the same expectation), thus it has no impact on the long-run fraction of alarms taken over if the times are as described in part (b).

Exercise 3.

(a) For the load it holds that $\rho = \frac{3}{4b} \times b = \frac{3}{4}$. Thus,

$$\mathbb{E}W^q = \frac{1}{2}(1 + c_B^2)\mathbb{E}B\frac{\rho}{1-\rho} = \frac{1}{2}b\frac{3/4}{1-3/4} = \frac{3}{2}b.$$

(b) Using Little's law gives

$$\mathbb{E}L^q = \lambda \mathbb{E}W^q = \frac{3}{4b} \times \frac{3}{2}b = \frac{9}{8}.$$

Make a sketch ($\mathbb{E}W^q$ increases linearly in b and $\mathbb{E}L^q$ is constant). The load does not depend on b, explaining that $\mathbb{E}L^q$ is constant. When b changes, only the time scale changes (the speed at which events occur in the system), thus $\mathbb{E}W^q$ changes linearly.

(c) First, determine the first 2 moments of B:

$$\begin{aligned}\mathbb{E}B &= \frac{1}{2} \times \frac{1}{2}b + \frac{1}{2} \times \frac{3}{2}b = b \\ \mathbb{E}B^2 &= \frac{1}{2} \times \left(\frac{b}{2}\right)^2 + \frac{1}{2} \times \left(\frac{3b}{2}\right)^2 = \frac{5}{4}b^2\end{aligned}$$

Then, the expected waiting time is

$$\mathbb{E}W^q = \frac{\mathbb{E}B^2}{2\mathbb{E}B}\frac{\rho}{1-\rho} = \frac{5b^2/4}{2b}\frac{3/4}{1-3/4} = \frac{15}{8}b.$$

For the expected sojourn time, we have

$$\mathbb{E}W = \mathbb{E}W^q + \mathbb{E}B = \frac{15}{8}b + b = \frac{23}{8}b.$$

Exercise 4.

(a) The time until the next service completion, *given* that both servers are occupied, is $\exp(2\mu)$ distributed, hence

$$\mathbb{P}(R > t) = e^{-2\mu t}.$$

(b) The arrival relation is

$$\mathbb{E}[W^q] = \underbrace{\mathbb{E}[L^q]}_{\text{I}} \times \underbrace{\frac{1}{2\mu}}_{\text{II}} + \underbrace{\Pi_W}_{\text{III}} \times \underbrace{\frac{1}{2\mu}}_{\text{IV}}.$$

with interpretation:

I	$\mathbb{E}$ # customers in the queue
II	$\mathbb{E}$ time to serve 1 customer (super-server)
III	probability of waiting
IV	residual time until first service completion

Applying Little's law ($\mathbb{E}L^q = \lambda \mathbb{E}W^q$):

$$\mathbb{E}W^q = \mathbb{E}W^q \times \frac{\lambda}{2\mu} + \Pi_W \frac{1}{2\mu}.$$

Isolating $\mathbb{E}W^q$ gives

$$\mathbb{E}W^q = \Pi_W \frac{1}{2\mu(1-\rho)} \qquad \text{with } \rho = \frac{\lambda}{2\mu}.$$

(c) The arrival relation for LCFS-NP is

$$\mathbb{E}[W^q] = \underbrace{\Pi_W}_{\text{I}} \left(\underbrace{\frac{1}{2\mu}}_{\text{II}} + \underbrace{\frac{1}{2\mu}\lambda}_{\text{III}} \underbrace{\mathbb{E}[BP]}_{\text{IV}} \right),$$

with interpretation

I	probability of waiting (both servers occupied)
II	residual time R until first service completion
III	$\mathbb{E}$ # arrivals during R
IV	each customers has an extended service time, similar to a busy period (expected time until server is 'available' again to serve customers already waiting)