Midterm Stochastic Modeling (400646)

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October 20, 2015, 12:00 - 14:00 hours

This (midterm) exam consist of four exercises. The use of books or a graphical calculator is not allowed.

At each part it is indicated between square brackets how many points can be achieved for the corresponding part. The grade for the midterm is given by $p/3 + 1$ where p is the total number of points earned by you. Please include your name and student number on all papers and motivate all your answers clearly.

Good luck!

Exercise 1

Consider a discrete-time Markov chain with state space $\{1, 2, 3, 4, 5, 6\}$ and transition matrix

$$\begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{4} & 0 & 0 & \frac{3}{4} & 0 \end{pmatrix}$$

(a) [1 pt.] Draw the state diagram of possible transitions and determine the classes of communicating states.

(b) [2 pt.] Determine the long-run fraction of time that the Markov chain remains in each of the states.

(c) [2 pt.] What is the probability that the Markov chain will *never* make a direct transition from state 1 to state 3, when the initial state is 1?

(d) [2 pt.] What is the expected number of transitions to reach state 2, starting in state 1?

Exercise 2

Rijkswaterstaat performs traffic measurements on the ring road around Utrecht.

(a) [3 pt.] It is found that three out of four trucks on the road is followed by a car, whereas only one out of five cars is followed by a truck. What is the fraction of trucks on the road?

(b) [2 pt.] It is also found that the number of cars passing a measurement point follows a Poisson process with rate λ_1 per hour. What is the probability that during the next minute at least 5 cars pass the measurement point (expressed in terms on the rate λ_1)?

(c) [2 pt.] The number of trucks that pass the measurement point follows a Poisson process with rate λ_2 per hour. It also turns out that 30% of the truck drivers does not adhere to the guidelines for taking rest. What is the probability that there are exactly three trucks with a driver adhering to the rest guidelines pass the measuring point before the first car passes the measurement point (expressed in terms of the rates)?

Exercise 3[1]

A small but popular art museum is open day and night and has a target capacity of D people. Physically, much more than D people fit in the museum, but the museum finds that such overcrowding has a negative impact on customer experience. If there are too many customers in the museum during a certain hour, the museum uses a penalty of B for each customer that is beyond the target capacity D. Access to the museum is strictly controlled. At the beginning of each hour, exactly M people are allowed to enter (from the long queue in front of the museum). The time that customers spend in the museum (the visit time) is random. These visit times of different customers are independent of each other and are exponentially distributed

[1] This exercise is rather complicated. We decided to modify the correction prescription based on this exercise.

with expectation $1/\mu$ hours. We are interested in the progress of the number of customers in the museum per hour of the day.

(a) [3 pt.] Define an appropriate discrete-time Markov chain and specify the one-step transition probabilities.

(b) [4 pt.] Determine the balance equations and determine the expected long-run costs per time unit in terms of the limiting distribution.

Exercise 4

Data packets arrive at a router according to a Poisson process with an average of 10 packets per millisecond. The router has a buffer of three packets; packets arriving when there are already three packets present at the router are rejected. The time it takes to process a data package is exponentially distributed with an expectation of 0.05 milliseconds. The router processes the packets based on 'first come, first served' (FCFS).

(a) [3 pt.] Calculate the expected number of data packets that is present in the buffer of the router.

(b) [3 pt.] If the router is now taken into use, what is the expected time until the buffer is full for the first time?