Midterm Stochastic Modeling (400646) - Solutions

The solutions are always provisionary

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October 20, 2015, 12:00 - 14:00 hours

Exercise 1

(a) [1 pt.] See Figure 1 for the state diagram. The classes of communicating states are $\{1, 4\}$ (transient), $\{3\}$ (transient), and $\{2, 5, 6\}$ absorbing.

(b) [2 pt.] We need to determine the occupancy time. The classes $\{1, 4\}$ and $\{3\}$ are transient, yielding that $\hat{\pi}_1 = \hat{\pi}_3 = \hat{\pi}_4 = 0$. We thus need to determine the occupancy distribution for the set $\{2, 5, 6\}$. The balance equations for this set are

$$\pi_2 = \frac{1}{4}\pi_6 + \frac{1}{2}\pi_5$$
$$\pi_5 = \frac{1}{2}\pi_2 + \frac{3}{4}\pi_6$$
$$\pi_6 = \frac{1}{2}\pi_2 + \frac{1}{2}\pi_5$$

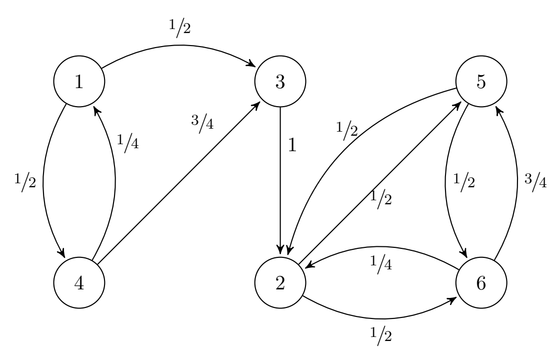

Figure 1: State diagram of the DTMC of exercise 1a.

Subtracting the third from the second equation gives $\pi_5 - \pi_6 = \frac{3}{4}\pi_6 - \frac{1}{2}\pi_5$. Rewriting this equation to express π_5 in terms of π_6 yields $\pi_5 = \frac{7}{6}\pi_6$. From the first equation, we now have $\pi_2 = \frac{5}{6}\pi_6$. Using normalization ($\pi_2 + \pi_5 + \pi_6 = 1$) gives $\pi_6(\frac{5}{6} + \frac{7}{6} + 1) = 1$, providing $\pi_6 = \frac{1}{3}$. Thus the occupancy distribution is $\hat{\pi} = (0, {}^5\!/_{18}, 0, 0, {}^7\!/_{18}, {}^1\!/_3)$.

(c) [2 pt.] Let p_i be the probability that the Markov chain will *never* make a direct transition from state 1 to state 3, given that the current state is i. We are then looking for p_1. Conditioning on the first transition gives the set of equations:

$$p_1 = \frac{1}{2} \times 0 + \frac{1}{2} \times p_4$$
$$p_4 = \frac{3}{4} \times 1 + \frac{1}{4} \times p_1$$

Substituting the second in the first equation provides $p_1 = {}^1\!/_2({}^3\!/_4 + p_1{}^1\!/_4)$. Hence, the required quantity is $p_1 = {}^3\!/_7$.

(d) [2 pt.] Let m_i be the expected number of transitions to reach state 2 given that the current state is i. We are looking for m_1. Conditioning on the next transition, we obtain the following set of equations:

$$m_1 = 1 + \frac{1}{2}m_3 + \frac{1}{2}m_4$$
$$m_4 = 1 + \frac{1}{4}m_1 + \frac{3}{4}m_3$$
$$m_3 = 1$$

Solving this set of equations yields $m_1 = {}^{19}\!/_7$ (as an intermediate step, one may also verify that $m_4 = {}^{17}\!/_7$).

Exercise 2

(a) [3 pt.] To determine the fraction of trucks, we define an appropriate discrete-time Markov chain (DTMC). Let X_n be the type of the nth vehicle. Then $\{X_n, n = 0, 1, \ldots\}$ is a DTMC on the state space $I = \{C, T\}$ (it is advised to draw a state diagram). The balance equation for state C is

$$\pi_C = \frac{4}{5}\pi_C + \frac{3}{4}\pi_T.$$

From normalization ($\pi_C + \pi_T = 1$), we have $\pi_T = 1 - \pi_C$. Substituting this in the equation above and solving for π_C yields $\pi_C = {}^{15}\!/_{19}$. So the fraction of trucks on the road is ${}^4\!/_{19}$.

(b) [2 pt.] Let $N_C(t)$ be the number of cars passing the measurment point during $[0, t)$ with t in minutes. The rate of this Poisson process is $\lambda_1/60$ per *minute*. Then

$$\mathbb{P}(N_C(1) \geq 5) = 1 - \mathbb{P}(N_C(1) \leq 4) = 1 - \sum_{k=0}^{4} e^{-\lambda_1/60} \frac{(\lambda_1/60)^k}{k!}.$$

(c) [2 pt.] The number of trucks with drivers that adhere to the guidelines and that pass the measurement point follows a Poisson process with rate $0.7\lambda_2$ per hour (due to thinning of a Poisson process). Thus, the times between two vehicles passing the measurement are exponential with rates λ_1 and $0.7\lambda_2$ per hour, for cars and trucks with drivers that adhere to the guidelines, respectively. For the required probability, the second exponential 'beats' the first 3 times, after which the first exponential 'beats' the second, i.e.,

$$\left(\frac{0.7\lambda_2}{0.7\lambda_2 + \lambda_1}\right)^3 \left(\frac{\lambda_1}{0.7\lambda_2 + \lambda_1}\right)$$

Exercise 3

(a) [3 pt.] Let X_n be the number of customers in the museum at the start of hour n (just before the admission of new customers). Then $\{X_n, n = 0, 1, \ldots\}$ is a DTMC on the state space $I = \{0, 1, \ldots\}$. For the one-step transition probabilities we observe the following. If there are i customers just before admission, then just after admission there are $i + M$ customers; the number of departures during the consecutive hour then follows a binomial distribution with parameters $i + M$ and $1 - e^{-\mu}$ (i.e. the probability that a customer leaves in one hour is $1 - e^{-\mu}$ due to the exponential distribution). Thus, for $j = 0, 1, \ldots, i + M$, we have one-step transition probabilities

$$\mathbb{P}(X_{n+1} = j \mid X_n = i) = \binom{i+M}{j} \left(e^{-\mu}\right)^j \left(1 - e^{-\mu}\right)^{i+M-j}$$

These probabilties are 0 for $j > i+M$, since there are at most M new customers during one hour.

(b) [4 pt.] The balance equations read

$$\pi_j = \sum_{k=0}^{\infty} \binom{k+M}{j} \left(e^{-\mu}\right)^j \left(1 - e^{-\mu}\right)^{k+M-j} \pi_k, \qquad \text{for } j = 0, 1, \ldots, M$$

and

$$\pi_j = \sum_{k=j-M}^{\infty} \binom{k+M}{j} \left(e^{-\mu}\right)^j \left(1-e^{-\mu}\right)^{k+M-j} \pi_k, \qquad \text{for } j = M+1, \ldots$$

Assuming that the penalty is charged based on the number of customers at the start of an hour, the expected long-run cost per hour is

$$B \times \sum_{k=D}^{\infty} (k-D)\pi_k$$

Exercise 4

Data packets arrive at a router according to a Poisson process with an average of 10 packets per millisecond. The router has a buffer of three packets; packets arriving when there are already three packets present at the router are rejected. The time it takes to process a data package is exponentially distributed with an expectation of 0.05 milliseconds. The router processes the packets based on 'first come, first served' (FCFS).

(a) [3 pt.] To answer the question, we formulate a continuous-time Markov chain (CTMC). Let $X(t)$ be the number of packets at the router at time t. Then $\{X(t), t \geq 0\}$ is a CTMC on state space $I = \{0, 1, 2, 3\}$. Let the time unit be in milliseconds, in which case $q_{i,i+1} = 10$ for $i = 0, 1, 2$ and $q_{i,i-1} = 20$ for $i = 1, 2, 3$. For convenience, draw a state diagram with transition rates!

Now, the balance equations are

$$\begin{aligned} 10p_0 &= 20p_1 \\ 30p_1 &= 10p_0 + 20p_2 \\ 30p_2 &= 10p_1 + 20p_3 \\ 20p_3 &= 10p_2 \end{aligned}$$

The last equation provides $p_3 = \frac{1}{2}p_2$. From the third equation, we now have $30p_2 = 10p_1 + 10p_2$, such that $p_2 = \frac{1}{2}p_1$. From the first equation it holds that $p_1 = \frac{1}{2}p_0$. Determining p_0 using normalization gives $p_0(1+1/2+(1/2)^2+(1/2)^3) = 1$; thus $p_0 = 8/15$. Now, the expected number of packets present in the buffer is

$$\begin{aligned}\mathbb{E}X &= \sum_{k=1}^{3} kp_k \\ &= 1 \times \frac{1}{2}\frac{8}{15} + 2 \times \left(\frac{1}{2}\right)^2 \frac{8}{15} + 3 \times \left(\frac{1}{2}\right)^3 \frac{8}{15} \\ &= \frac{4}{15} + \frac{4}{15} + \frac{3}{15} = \frac{11}{15}\end{aligned}$$

(b) [3 pt.] Let m_i denote the expected time until the buffer is full for the first time, given the current state is i. We are then looking for m_0. Conditioning on the next transition then provides the following set of equations:

$$\begin{aligned}m_0 &= \frac{1}{10} + m_1 \\ m_1 &= \frac{1}{30} + \frac{2}{3}m_0 + \frac{1}{3}m_2 \\ m_2 &= \frac{1}{30} + \frac{2}{3}m_1\end{aligned}$$

Solving this set of equations (e.g. by first substituting the third in the second equations, and later the second in the first equation) gives the solution $m_0 = {}^{11}/_{10}$.