X_400004 - Statistics
Solutions to the Extra Resit

9 July 2021

Below are answers to the exam questions. Some of these are slightly abbreviated, while others include extra comments. These are for your reference only but should inform the level of detail that is expected from your answers in the exam. Also keep in mind that there might be different ways to approach each question. If you find typos and or omissions, please report them to the lecturer so they can be corrected.

Prob.I: Suppose that X is a Poisson random variable with unknown rate $\lambda > 0$. The following are 10 independent observations that were taken from such a distribution: $\{3, 0, 2, 1, 3, 2, 1, 0, 2, 1\}$.

3 pts (a) Find the method of moments estimator of λ. What is the method of moments estimate?

Solution: We have that $\mathbb{E}X = \lambda$. To get the MME we solve $\lambda = \bar{X}$, which immediately gives us the estimator $\hat{\lambda} = \bar{X}$. From the data $\bar{X} = 15/10$, which plugging into the estimator leads to the estimate 1.5 for λ.

6 pts (b) What is the maximum likelihood estimator of λ? What is the maximum likelihood estimate?

Solution: Let n represent the number of observations. The likelihood function is

$$L(\lambda) = \prod_{i=1}^{n} \frac{1}{X_i!} e^{-\lambda} \lambda^{X_i} = e^{-n\lambda} \lambda^{\sum_{i=1}^{n} X_i} \prod_{i=1}^{n} \frac{1}{X_i!}.$$

To maximise this function we can take the logarithm and then derivative with respect to λ to get

$$\frac{d \log L(\lambda)}{d\lambda} = \frac{d}{d\lambda} \left(-n\lambda + \sum_{i=1}^{n} X_i \log \lambda - \sum_{i=1}^{n} \log(X_i!) \right) = -n + \frac{1}{\lambda} \sum_{i=1}^{n} X_i.$$

Solving for 0 gives

$$\frac{1}{\lambda} \sum_{i=1}^{n} X_i = n \Leftrightarrow \sum_{i=1}^{n} X_i = n\lambda \Leftrightarrow \hat{\lambda} = \bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

So the maximum likelihood estimate is also 1.5.

6 pts (c) If the prior distribution on λ is $Gamma(1, 2)$, what is the posterior distribution? What is the estimate that you obtain from the posterior expectation?

Solution: From the previous question we know that the likelihood satisfies

$$L(\theta) \propto e^{-n\lambda} \lambda^{\sum_{i=1}^{n} X_i}.$$

Multiplying this with the prior $\pi(\lambda) \propto \lambda^{1-1}e^{-2\lambda} = e^{-2\lambda}$ we get that the posterior is proportional to

$$e^{-n\lambda}\lambda^{\sum_{i=1}^n X_i}e^{-2\lambda} = \lambda^{1+\sum_{i=1}^n X_i - 1}e^{-(n+2)\lambda}.$$

We recognise this as the density of a $Gamma(1 + \sum_{i=1}^n X_i, n+2)$ distribution. From here we see that the posterior expectation is $(1 + \sum_{i=1}^n X_i)/(n+2)$ leading to the estimate $16/12 \approx 1.33$.

8 pts (d) Consider the estimators $\tilde{\lambda} = (a + \sum_{i=1}^n X_i)/(n+b)$ for $a, b \geq 0$. Find the bias and the variance of $\tilde{\lambda}$ and in particular of the three estimators that you found. (So the MME, MLE, and BE.) What can you conclude?

Solution: All of the estimators are of the form $\tilde{\lambda}$ for different a, b. We have

$$\mathbb{E}\tilde{\lambda} = \mathbb{E}\frac{a + \sum_{i=1}^n X_i}{n+b} = \frac{a + \mathbb{E}\sum_{i=1}^n X_i}{n+b} = \frac{a + \sum_{i=1}^n \mathbb{E}X_i}{n+b} = \frac{n\lambda + a}{n+b},$$

where we use the fact that the expectation is linear, and the distribution of the X_i is $Poisson(\lambda)$. The bias is therefore

$$\frac{n\lambda + a}{n+b} - \lambda = \frac{a - b\lambda}{n+b}.$$

Next we compute the variance,

$$\mathbb{V}\tilde{\lambda} = \mathbb{V}\frac{a + \sum_{i=1}^n X_i}{n+b} = \frac{\mathbb{V}(a + \sum_{i=1}^n X_i)}{(n+b)^2} = \frac{\sum_{i=1}^n \mathbb{V}X_i}{(n+b)^2} = \frac{n\lambda}{(n+b)^2}.$$

From this, the MME and MLE ($a = b = 0$) have bias and variance respectively

$$0, \qquad \text{and} \qquad \frac{\lambda}{n} = \frac{\lambda}{10},$$

and the BE ($a = 1, b = 2$) has bias and variance respectively

$$\frac{1-2\lambda}{n+2} = \frac{1-2\lambda}{12}, \qquad \text{and} \qquad \frac{n\lambda}{(n+2)^2} = \frac{\lambda}{14.4}.$$

So the BE has (in general) higher bias than the MME, MLE, but it always has smaller variance.

Hint: If $X \sim Poisson(\lambda)$, then $p(x) = e^{-\lambda}\lambda^x/x!$, so that $\mathbb{E}X = \lambda$ and $\mathbb{V}X = \lambda$, and if $Y \sim Gamma(\alpha, \beta)$, then $f(y) = \beta^\alpha y^{\alpha-1}e^{-\beta x}/\Gamma(\alpha)$, so that $\mathbb{E}Y = \alpha/\beta$ and $\mathbb{V}Y = \alpha/\beta^2$.

Prob.II: Suppose that you are managing a website. Let $X_1, \dots, X_n$ be a random sample from the Poisson distribution with unknown parameter $\lambda > 0$. Each observation X_i corresponds to the number of visitors to your website in a given day. You are interested in the value of λ, the expected number of visitors per day.

6 pts (a) Describe a near-pivot T for λ. Justify how you arrived at T, and don't forget to mention the (approximate) distribution of the pivot.

Solution: Since we have a random sample from a Poisson distribution, the Central Limit Theorem tells us that if n is large enough, then

$$\frac{\sum_{i=1}^n X_i - \mathbb{E}\sum_{i=1}^n X_i}{\sqrt{\mathbb{V}\sum_{i=1}^n X_i}} = \frac{\bar{X} - \lambda}{\sqrt{\lambda/n}} = \frac{n\bar{X} - n\lambda}{\sqrt{n\lambda}} = \sqrt{n}\frac{\bar{X} - \lambda}{\sqrt{\lambda}} \approx N(0,1).$$

The above is already a near-pivot but the pivot can be simplified by noting that the (weak/strong) law of large numbers implies that $\bar{X}$ converges in probability to $\mathbb{E}X = \lambda$ so that we are also allowed to claim that

$$T = \frac{n\bar{X} - n\lambda}{\sqrt{n\bar{X}}} \approx N(0,1).$$

9 pts (b) Use the near-pivot from the previous question to derive an approximate confidence interval of level 0.95 for λ. Don't just present the final result: show how you go from (near-)pivot to confidence interval.

Solution: Since T is a near-pivot and is approximately standard normal distributed, we know that since $z_{0.025} = -z_{0.975}$

$$0.95 \approx \mathbb{P}\left(-z_{0.975} \leq \frac{n\bar{X} - n\lambda}{\sqrt{n\bar{X}}} \leq z_{0.975}\right)$$
$$= \cdots = \mathbb{P}\left(\bar{X} - \frac{\sqrt{n\bar{X}}}{n} z_{0.975} \leq \lambda \leq \bar{X} + \frac{\sqrt{n\bar{X}}}{n} z_{0.975}\right),$$

which leads to a confidence interval for p of level 0.95:

$$\left[\bar{X} - \frac{\sqrt{n\bar{X}}}{n} z_{0.975},\ \bar{X} + \frac{\sqrt{n\bar{X}}}{n} z_{0.975}\right].$$

4 pts (c) Suppose that you collect data for $n = 10$ days and obtain the confidence interval $[27, 42]$ for λ. Considering how the interval was derived, is it correct to say that this confidence interval contains λ with probability 0.95? Justify your answer.

Solution: The statement is clearly incorrect. The interval $[27, 42]$ is not random and so either contains λ or not, but it is not random and so cannot contain λ with any pre-specified probability.

8 pts (d) Suppose that you are not happy with the large amount of uncertainty in the confidence interval that you got – you find the confidence interval too wide. We saw in class that the length of a confidence interval usually decreases as you increase the sample size. During how many days would you have to collect data to be sure that the length of the confidence interval does not exceed 10 people? Do you envision any problems with implementing this in practice?

Solution: The length of the CI is

$$\bar{X} + \frac{\sqrt{n\bar{X}}}{n} z_{0.975} - \left(\bar{X} - \frac{\sqrt{n\bar{X}}}{n} z_{0.975} \right) = 2\frac{\sqrt{n\bar{X}}}{n} z_{0.975}.$$

For this to be smaller than 10 we need

$$2\frac{\sqrt{n\bar{X}}}{n} z_{0.975} \leq 10 \Leftrightarrow \sqrt{\frac{\bar{X}}{n}} \leq \frac{5}{z_{0.975}} \Leftrightarrow \frac{\bar{X}}{n} \leq \frac{25}{z_{0.975}^2} \Leftrightarrow z_{0.975}^2 \frac{\bar{X}}{25} \leq n.$$

The problem here is that $\bar{X}$ also depends on the sample size so we cannot determine n ahead of time.

Prob.III: Atlanta International Airport (ATL) has been voted many times as the world's most efficient airport. When looking over some historical data about waiting times for luggage at this airport, you come across the following statistical problem. Suppose that the amount of time (in minutes) that a passenger has to wait for their bag is well modelled by an exponential distribution with parameter λ (and therefore expectation $1/\lambda$).

Suppose that you have observations from n passengers, denoted by $X_1, \ldots, X_n$, that can be assumed to be an i.i.d. sample from an exponential distribution with parameter λ. We would like to conduct the following hypothesis test

$$H_0 : \lambda = 0.25 \quad \text{against} \quad H_1 : \lambda < 0.25.$$

In other words, is the average time a passenger has to wait for their luggage $1/0.25 = 4$ minutes, or is it longer than that? A possible test statistic to consider is $Y = \sum_{i=1}^{n} X_i$. From your knowledge of probability, you know that Y has an Erlang distribution with parameters n and λ.

Consider a test procedure that rejects the null hypothesis if $Y \geq c_\alpha$, where $c_\alpha > 0$ is a critical value that must be chosen depending on the desired significance level. **For the rest of the question consider the case $n = 3$.**

9 pts (a) Say that you take $c_\alpha = 32$. What is the type I error of this test?

Solution: The type I error corresponds to rejecting the null hypothesis when the null hypothesis is actually true. Therefore, the type I error is given by

$$\begin{aligned}
P_{H_0}(\textbf{reject } H_0) &= P_{\lambda=0.25}\left(Y \geq c_\alpha\right) = P_{\lambda=0.25}\left(Y \geq 32\right) \\
&= 1 - P_{\lambda=0.25}\left(Y < 32\right) = 1 - \left(1 - e^{-32\lambda} \sum_{k=0}^{n-1} \frac{(32\lambda)^k}{k!}\right) \\
&= e^{-32\lambda}\left(1 + 32\lambda + \frac{(32\lambda)^2}{2}\right) = e^{-8}\left(1 + 8 + 8^2/2\right) \approx 0.01375,
\end{aligned}$$

where we use the fact that $n = 3$ and set $\lambda = 0.25 = 1/4$.

9 pts (b) At a less efficient airport we expect the passenger to have to wait on average 16 minutes, meaning $\lambda = 1/16$. What is the power of the test in the previous question (where $c_\alpha = 32$) in this case?

Solution: The power of a test is the probability of rejecting the null hypothesis when it should indeed be rejected. In this case $\lambda = 1/16$ so the power of this test is given by

$$\begin{aligned}
P_{H_1}(\textbf{reject } H_0) &= P_{\lambda=1/16}\left(Y \geq c_\alpha\right) = P_{\lambda=1/16}\left(Y \geq 32\right) = 1 - P_{\lambda=1/16}\left(Y < 32\right) \\
&= 1 - \left(1 - e^{-32\lambda} \sum_{k=0}^{n-1} \frac{(32\lambda)^k}{k!}\right) = e^{-32\lambda}\left(1 + 32\lambda + \frac{(32\lambda)^2}{2}\right) \\
&= e^{-2}\left(1 + 2 + 2^2/2\right) \approx 0.6767.
\end{aligned}$$

9 pts (c) An experiment was conducted and a total waiting time (for the $n = 3$ passengers) of $y = 28$ minutes was recorded. Compute the p-value of this test. Would you reject the hull hypothesis at significance level $\alpha = 0.05$? Carefully justify your answer.

Solution: The p-value of a test is the smallest significance level for which the null hypothesis is rejected. So this corresponds to the largest value of c_α for which the test will reject H_0. Clearly, we must take $c_\alpha = 28$ since is $c_\alpha > 28$ we don't reject. The type I error of this test (i.e., the p-value) is then

$$\begin{aligned}\textbf{p-value} &= P_{\lambda=0.25}(Y \geq 28) = P_{\lambda=0.25}(Y \geq 28) = 1 - P_{\lambda=0.25}(Y < 28) \\ &= 1 - \left(1 - e^{-28\lambda}\sum_{k=0}^{n-1}\frac{(28\lambda)^k}{k!}\right) = e^{-28\lambda}\left(1 + 28\lambda + \frac{(28\lambda)^2}{2}\right) \\ &= e^{-7}\left(1 + 7 + 7^2/2\right) \approx 0.02964.\end{aligned}$$

Since the p-value is smaller than 0.05 we would reject the null hypothesis at significantly level $\alpha = 0.05$.

Hint: An Erlang random variable Y with parameters n and λ has density

$$f_Y(y) = \begin{cases} e^{-\lambda y}\frac{\lambda^n y^{n-1}}{(n-1)!} & \text{if } y \geq 0 \\ 0 & \text{otherwise} \end{cases},$$

and cumulative distribution function

$$F_Y(y) = P(Y \leq y) = \begin{cases} 1 - e^{-\lambda y}\sum_{k=0}^{n-1}\frac{(\lambda y)^k}{k!} & \text{if } y \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

Prob.IV: Consider the following situation. Someone enjoys preparing home made pizzas but the downside of making the pizzas by hand is that they don't always come out the same size and so it is difficult to know how long they should cook for.

Someone went through the trouble of watching several pizzas like a hawk while they cook until they looked just right. The table below contains the data that was collected. Each row corresponds to a different pizza, and for each pizza the weight (which seems the most relevant quantity to determine the ideal cooking time) and the ideal cooking time were recorded.

Weight (g)	Cooking time (minutes)
57	19.0
88	19.5
89	23.5
73	17.5
91	21.0
100	20.0
70	21.5
109	22.0
101	25.0
91	20.5
79	22.0
96	23.5
82	22.5
101	23.5

A simple linear regression analysis is to be conducted where the cooking time is taken as the response variable and the weight of the pizza as the predictor. The model can be useful to predict ahead of time what the ideal cooking time for the pizza is, based on its weight. **Have a look at Appendix A before you start solving this problem.**

3 pts (a) Write the assumed **model equation** for the relation between the pizza weight and the cooking time using α to denote the intercept and β the slope. Suppose that you fit the model and get the plot in Figure 2 for the resulting residuals. Is it reasonable to assume normally distributed errors?

Solution: Let y_i denote the cooking time on the i-th day, and x_i denote the pizza weight. In the regression model we assume that

$$y_i = \alpha + \beta x_i + \epsilon_i,$$

where $\alpha, \beta \in \mathbb{R}$ are unknown parameters and ϵ_i are zero mean independent random errors. From Figure 2 we see that most points lie relatively close to a straight line in the normal QQ plot of the residuals. Therefore the normality assumption is somewhat reasonable.

8 pts (b) Estimate the model parameters α and β from the data, as well as σ^2, the variance of the noise.

Solution: We have that

$$
\begin{aligned}
S_{xx} &= \sum_{i=1}^{n} x_i^2 - n(\bar{x})^2 = 2631.214, \\
S_{yy} &= \sum_{i=1}^{n} y_i^2 - n(\bar{y})^2 = 55.5, \\
S_{xy} &= \sum_{i=1}^{n} x_i y_i - n\bar{y}\bar{x} = 204.5, \\
\hat{\beta} &= S_{xy}/S_{xx} = 0.07772077, \\
\hat{\alpha} &= \bar{y} - \hat{\beta}\bar{x} = 14.68833, \\
\hat{\sigma}^2 &= S_{yy}/n - \hat{\beta}^2 S_{xx}/n = 2.829007.
\end{aligned}
$$

3 pts (c) Suppose that you prepared a pizza and it weighs 93g. Give an estimate for the ideal cooking time (according to your model) for that pizza.

Solution: Let $x_0 = 93$. A point estimate for the cooking time is

$$\hat{\mu} = \hat{\alpha} + 93 \times \hat{\beta} = 14.68833 + 93 \times 0.07772 = 21.92 \text{ **minutes**}.$$

9 pts (d) You are a bit cautious since there could be other factors at play so that the estimate from (c) might be off and you don't want to under- or over-cook your pizza. Use your knowledge of regression models to give a two sided **prediction interval** for the ideal cooking time in the scenario in (c) (use $\alpha = 0.05$).

Solution: We need to compute a two-sided prediction interval for the cooking time. The end-points of this prediction interval are

$$\hat{\mu} \pm t_{1-\alpha/2;14-2} \sqrt{\hat{\sigma}^2 \left(1 + \frac{1}{n} + \frac{(x_0 - \bar{x})^2}{S_{xx}}\right)}.$$

Note that $t_{0.975;12} = 2.179$ (looked up in the 0.025 column of the table of t-probabilites.) Plugging in everything, we conclude the prediction interval is $[18.10346, 25.72926]$ minutes.

A Pizza Data

From the dataset we get $\bar{x} = 87.64286$, $\bar{y} = 21.5$, $\sum_{i=1}^{n} x_i^2 = 110169$, $\sum_{i=1}^{n} y_i^2 = 6527$, $\sum_{i=1}^{n} x_i y_i = 26585$.

Below follows a normal QQ-plot of the residuals of the model.

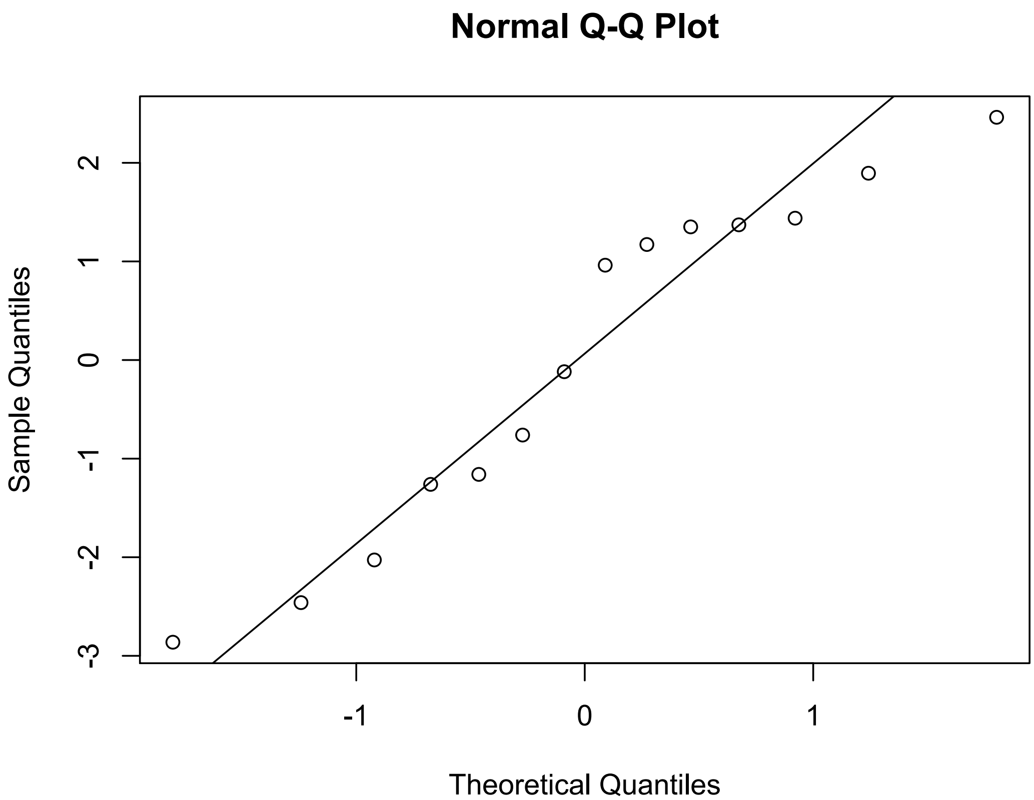

Figure 1: Normal QQ plot of the residuals of the regression model.

B Quantiles and cumulative probability tables

Below you can find a table of probabilities for the t-distribution and for the standard normal distribution, respectively. Please read the caption of the tables carefully.

$\downarrow \nu/\alpha \rightarrow$	0.3	0.2	0.15	0.1	0.05	0.025	0.02	0.01	0.005	0.0025	0.001
1	0.727	1.376	1.963	3.078	6.314	12.71	15.90	31.82	63.66	127.3	318.3
2	0.617	1.061	1.386	1.886	2.920	4.303	4.849	6.965	9.925	14.10	22.33
3	0.584	0.978	1.250	1.638	2.353	3.182	3.482	4.541	5.841	7.453	10.215
4	0.569	0.941	1.190	1.533	2.132	**2.776**	2.999	3.747	4.604	5.598	7.173
5	0.559	0.920	1.156	1.476	2.015	2.571	2.757	3.365	4.032	4.773	5.893
6	0.553	0.906	1.134	1.440	1.943	2.447	2.612	3.143	3.707	4.317	5.208
7	0.549	0.896	1.119	1.415	1.895	2.365	2.517	2.998	3.499	4.029	4.785
8	0.546	0.889	1.108	1.397	1.860	2.306	2.449	2.896	3.355	3.833	4.501
9	0.543	0.883	1.100	1.383	1.833	2.262	2.398	2.821	3.250	3.690	4.297
10	0.542	0.879	1.093	1.372	1.812	2.228	2.359	2.764	3.169	3.581	4.144
11	0.540	0.876	1.088	1.363	1.796	2.201	2.328	2.718	3.106	3.497	4.025
12	0.539	0.873	1.083	1.356	1.782	2.179	2.303	2.681	3.055	3.428	3.930
13	0.538	0.870	1.079	1.350	1.771	2.160	2.282	2.650	3.012	3.372	3.852
14	0.537	0.868	1.076	1.345	1.761	2.145	2.264	2.624	2.977	3.326	3.787
15	0.536	0.866	1.074	1.341	1.753	2.131	2.249	2.602	2.947	3.286	3.733
16	0.535	0.865	1.071	1.337	1.746	2.120	2.235	2.583	2.921	3.252	3.686
17	0.534	0.863	1.069	1.333	1.740	2.110	2.224	2.567	2.898	3.222	3.646
18	0.534	0.862	1.067	1.330	1.734	2.101	2.214	2.552	2.878	3.197	3.610
19	0.533	0.861	1.066	1.328	1.729	2.093	2.205	2.539	2.861	3.174	3.579
20	0.533	0.860	1.064	1.325	1.725	2.086	2.197	2.528	2.845	3.153	3.552
21	0.532	0.859	1.063	1.323	1.721	2.080	2.189	2.518	2.831	3.135	3.527
22	0.532	0.858	1.061	1.321	1.717	2.074	2.183	2.508	2.819	3.119	3.505
23	0.532	0.858	1.060	1.319	1.714	2.069	2.177	2.500	2.807	3.104	3.485
24	0.531	0.857	1.059	1.318	1.711	2.064	2.172	2.492	2.797	3.091	3.467
25	0.531	0.856	1.058	1.316	1.708	2.060	2.167	2.485	2.787	3.078	3.450
26	0.531	0.856	1.058	1.315	1.706	2.056	2.162	2.479	2.779	3.067	3.435
27	0.531	0.855	1.057	1.314	1.703	2.052	2.158	2.473	2.771	3.057	3.421
28	0.530	0.855	1.056	1.313	1.701	2.048	2.154	2.467	2.763	3.047	3.408
29	0.530	0.854	1.055	1.311	1.699	2.045	2.150	2.462	2.756	3.038	3.396
∞	0.524	0.842	1.036	1.282	1.645	1.960	2.054	2.326	2.576	2.807	3.090

Figure 2: Right quantiles of the student t-distribution with ν degrees of freedom. Example: if X is a student t distributed random variable, with $\nu = 4$ degrees of freedom then $P(X \geq 2.776) = 0.025$. **The entries of the table are therefore $t_{\nu,1-\alpha}$.**

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	**0.9115**	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Figure 3: Cumulative probabilities of the standard normal distribution. The rows correspond to the number rounded down to the closest decimal, and columns represent the second decimal. Example: if Z is a standard normal random variable then $P(Z \leq 1.35) = 0.9115$. You look this number up in the row corresponding to 1.3, and in the 6th column (corresponding to 0.05.) Note that for $z < 0$ we have $P(Z \leq z) = 1 - P(Z \leq -z)$. So for example $P(Z \leq -1.35) = 1 - 0.9115 = 0.0885$.