X_400004 - Statistics
Solutions to the Course Resit

9 February 2021

Below are answers to the exam questions. Some of these are slightly abbreviated, while others include extra comments. These are for your reference only but should inform the level of detail that is expected from your answers in the exam. Also keep in mind that there might be different ways to approach each question. If you find typos and or omissions, please report them to the lecturer so they can be corrected.

Prob.I: Suppose that X is a discrete random variable that with the following distribution

$$\mathbb{P}(X=0)=\frac{2}{3}\theta, \quad \mathbb{P}(X=1)=\frac{1}{3}\theta, \quad \mathbb{P}(X=2)=\frac{2}{3}(1-\theta), \quad \mathbb{P}(X=3)=\frac{1}{3}(1-\theta),$$

where $0 \leq \theta \leq 1$ is an unknown parameter. The following 10 independent observations were taken from such a distribution: $\{3,0,2,1,3,2,1,0,2,1\}$.

6 pts (a) Find the method of moments estimator of θ. What is the method of moments estimate?

Solution: We have that

$$\mathbb{E}X = 0\times\frac{2}{3}\theta + 1\times\frac{1}{3}\theta + 2\times\frac{2}{3}(1-\theta) + 3\times\frac{1}{3}(1-\theta) = \frac{\theta}{3}+\frac{4}{3}-\frac{4\theta}{3}+1-\theta = \frac{7-6\theta}{3}.$$

To get the MME we solve $(7-6\theta)/3=\bar{X}$, which gives us the estimator $\hat{\theta}=(7-3\bar{X})/6$. From the data $\bar{X}=15/10$, which plugging into the estimator leads to the estimate $5/12$.

6 pts (b) What is the maximum likelihood estimate of θ?

Solution: The likelihood of one observation is

$$\theta \mapsto \frac{2}{3}\theta 1_{\{X=0\}} + \frac{1}{3}\theta 1_{\{X=1\}} + \frac{2}{3}(1-\theta)1_{\{X=2\}} + \frac{1}{3}(1-\theta)1_{\{X=3\}},$$

which, combined with the fact that we observed $\{3,0,2,\ldots,1\}$ leads to the likelihood,

$$L(\theta) = \frac{1}{3}(1-\theta)\times\frac{2}{3}\theta\times\frac{2}{3}(1-\theta)\times\cdots\times\frac{1}{3}\theta = \left(\frac{2}{3}\theta\right)^2\times\left(\frac{1}{3}\theta\right)^3\times\left(\frac{2}{3}(1-\theta)\right)^3\times\left(\frac{1}{3}(1-\theta)\right)^2$$
$$= \left(\frac{2}{9}\theta(1-\theta)\right)^5.$$

From here it is already clear that this function is maximised by $\tilde{\theta}=1/2$, but this also follows from noting that the log-likelihood is

$$\ell(\theta) = 5\log\frac{2}{9} + 5\log\theta + 5\log(1-\theta);$$

taking derivatives and setting to zero we get

$$\frac{d\ell(\theta)}{d\theta} = \frac{5}{\theta} - \frac{5}{1-\theta} = 0 \Leftrightarrow 1 - \theta = \theta \Leftrightarrow \theta = 1/2,$$

which is our maximum likelihood estimate.

6 pts (c) If the prior distribution on θ is $U[0,1]$, what is the posterior density?

Solution: From the previous question we know that the likelihood is

$$L(\theta) = \left(\frac{2}{9}\theta(1-\theta)\right)^5.$$

multiplying this with the prior $\pi(\theta) = 1_{\{0\leq\theta<1\}}$ **we get that the posterior is proportional to**

$$\pi(\theta \mid 3, 0, 2, \ldots, 1) \propto \theta^5(1-\theta)^5 1_{\{0\leq\theta<1\}}.$$

We recognise this as the density of a Beta(6,6) distribution.

3 pts (d) Sketch the posterior. What is the mode of the posterior?

Solution: The mode of the posterior (the MAP estimator) is just 1/2 and coincides with the maximum likelihood estimator since we used a uniform prior. A sketch of the posterior density is depicted in Figure 1.

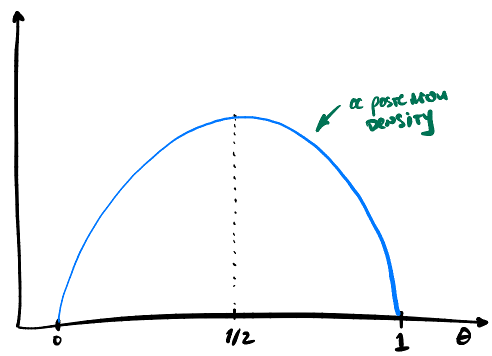

Figure 1: Sketch of the posterior density.

Prob.II: Suppose that you are managing an email mailing list for a website. Let $X_1, \ldots, X_n$ be a random sample from the Bernoulli distribution with unknown parameter $p \in [0,1]$. Each observation X_i corresponds to a different person that got the mailing list email. Suppose that $X_i = 1$ if the person opened the email, and $X_i = 0$, otherwise. You are interested in the value of p, the probability that someone who gets the mailing list actually opens it. (The other parameter in the model, n, is not specified but you always know what it is.)

6 pts (a) Describe a near-pivot T for p. Justify how you arrived at T, and don't forget to mention the (approximate) distribution of the pivot.

Solution: Let $\hat{p} = \bar{X}$ be the maximum likelihood estimator for p. The central limit tells us that if n is large enough, then

$$\frac{\bar{X} - \mathbb{E}\bar{X}}{\sqrt{\mathbb{V}\bar{X}}} = \frac{\bar{X} - p}{\sqrt{p(1-p)/n}} = \frac{n\bar{X} - np}{\sqrt{np(1-p)}} \approx N(0,1).$$

The above is already a near-pivot but the pivot can be simplified by noting that the (weak/strong) law of large numbers implies that $\hat{p}$ converges in probability to p so that we are also allowed to claim that

$$T = \frac{n\bar{X} - np}{\sqrt{n\hat{p}(1-\hat{p})}} \approx N(0,1).$$

9 pts (b) Use the near-pivot from the previous question to derive a confidence interval of level 0.95 for p. Don't just present the final result: show how you go from (near-)pivot to confidence interval.

Solution: Since T is a near-pivot and is approximately standard normal distributed, we know that since $z_{0.025} = -z_{0.975}$

$$0.95 \approx \mathbb{P}\left(-z_{0.975} \leq \frac{n\bar{X} - np}{\sqrt{n\hat{p}(1-\hat{p})}} \leq z_{0.975}\right) = \cdots =$$

$$= \mathbb{P}\left(\bar{X} - \frac{\sqrt{n\hat{p}(1-\hat{p})}}{n} z_{0.975} \leq p \leq \bar{X} + \frac{\sqrt{n\hat{p}(1-\hat{p})}}{n} z_{0.975}\right),$$

which leads to a confidence interval for p of level 0.95:

$$\left[\bar{X} - \frac{\sqrt{n\hat{p}(1-\hat{p})}}{n} z_{0.975},\ \bar{X} + \frac{\sqrt{n\hat{p}(1-\hat{p})}}{n} z_{0.975}\right].$$

6 pts (c) Suppose now that the last issue of the mailing list was sent to $n = 27$ subscribers and that, out of these, only 2 opened the email. Based on the confidence interval for p that you found, what would be your most optimistic guess as to how large p is? (There are tables at the end of the exam questionnaire in Appendix B that you may need to consult to answer this question.)

Solution: Using this information we see that $n = 27$, $\hat{p} = 2/27 \approx 0.074$, and from Appendix B we see that $z_{0.975} \approx 1.96$ (read this in the row labeled 1.9 and column

labeled 0.06) . The upper bound of the confidence interval gives us our most optimistic guess on how large p can be:

$$\bar{X} + \frac{\sqrt{n\hat{p}(1-\hat{p})}}{n} z_{0.975} = \frac{2}{27} + \frac{\sqrt{27 \times (2/27) \times (1 - 2/27)}}{27} 1.96 \approx 0.173,$$

which is not very high.

8 pts (d) Suppose that you are not happy with the large amount of uncertainty in the confidence interval that you got – you find the confidence interval too wide. In class, we saw that increasing the sample size n typically reduces uncertainty. How many emails would you have to send to be fairly confident that the estimate $\bar{X}$ of p is not off by more that 0.005? (You can still use $\bar{X}$ as your best guess for p.)

Solution: So that $|\hat{p} - p| = |\bar{X} - p|$ is no more than 0.005 we need to ensure that

$$\frac{\sqrt{n\hat{p}(1-\hat{p})}}{n} z_{0.975} = \sqrt{\frac{\bar{X} \times (1 - \bar{X})}{n}} 1.96 \leq 0.005.$$

Solving for n we conclude that (using $\bar{X} = 2/27 \approx 0.074$)

$$n \geq \frac{1.96^2 \times \bar{X}}{0.005^2} \approx 11382.5,$$

so we conclude that we would need at least (approximately) 11383 observations.

Prob.III: When studying the browsing behaviour in websites you come across the following statistical problem: users accessing a webpage will take some amount of time until they click the button "continue". Somewhat simplistically, the amount of time (in seconds) a user takes to press the button is well modeled by an exponential distribution with parameter λ (and therefore expectation $1/\lambda$). Suppose you have observations from n users, denoted by $X_1, \ldots, X_n$, that can be assumed to be an i.i.d. sample from an exponential distribution with parameter λ.

We would like to conduct the following hypothesis test

$$H_0 : \lambda = 0.25 \quad \text{against} \quad H_1 : \lambda < 0.25 \ .$$

In other words, is the average time a user stays on the page $1/0.25 = 4$ seconds, or is it larger? A natural test statistic to consider is $Y = \sum_{i=1}^{n} X_i$. From your knowledge of probability you know that Y has an Erlang distribution with parameters n and λ.

Hint: An Erlang random variable Y with parameters n and λ has density

$$f_Y(y) = \begin{cases} e^{-\lambda y} \frac{\lambda^n y^{n-1}}{(n-1)!} & \text{if } y \geq 0 \\ 0 & \text{otherwise} \end{cases} ,$$

and cumulative distribution function

$$F_Y(y) = P(Y \leq y) = \begin{cases} 1 - e^{-\lambda y} \sum_{k=0}^{n-1} \frac{(\lambda y)^k}{k!} & \text{if } y \geq 0 \\ 0 & \text{otherwise} \end{cases} .$$

Consider a test procedure that rejects the null hypothesis if $Y \geq c_\alpha$, where $c_\alpha > 0$ must be chosen depending on the desired significance level. **For the rest of the question consider the case $n = 3$.**

9 pts (a) Say that you take $c_\alpha = 32$. What is the type I error of this test?

Solution: The type I error corresponds to rejecting the null hypothesis when the null hypothesis is actually true. Therefore, the type I error is given by

$$\begin{aligned} P_{H_0}(\textbf{reject } H_0) &= P_{\lambda=0.25}(Y \geq c_\alpha) = P_{\lambda=0.25}(Y \geq 32) \\ &= 1 - P_{\lambda=0.25}(Y < 32) = 1 - \left(1 - e^{-32\lambda} \sum_{k=0}^{n-1} \frac{(32\lambda)^k}{k!}\right) \\ &= e^{-32\lambda}\left(1 + 32\lambda + \frac{(32\lambda)^2}{2}\right) = e^{-8}\left(1 + 8 + 8^2/2\right) \approx 0.01375, \end{aligned}$$

where we use the fact that $n = 3$ and set $\lambda = 0.25 = 1/4$.

9 pts (b) For a poorly designed webpage we expect the users to spend on average 16 seconds, meaning $\lambda = 1/16$. What is the power of the test in the previous question (where $c_\alpha = 32$) in this case?

Solution: The power of a test is the probability of rejecting the null hypothesis when it should actually be rejected. In this case $\lambda = 1/16$ so the power of this test

is given by

$$
\begin{aligned}
P_{H_1}(\textbf{reject } H_0) &= P_{\lambda=1/16}(Y \geq c_\alpha) = P_{\lambda=1/16}(Y \geq 32) = 1 - P_{\lambda=1/16}(Y < 32) \\
&= 1 - \left(1 - e^{-32\lambda} \sum_{k=0}^{n-1} \frac{(32\lambda)^k}{k!}\right) = e^{-32\lambda}\left(1 + 32\lambda + \frac{(32\lambda)^2}{2}\right) \\
&= e^{-2}\left(1 + 2 + 2^2/2\right) \approx 0.6767.
\end{aligned}
$$

9 pts (c) An experiment was conducted and a total waiting time of $y = 28$ seconds was recorded. Compute the p-value of this test. Would you reject the hull hypothesis at significance level $\alpha = 0.05$? Carefully justify your answer.

Solution: The p-value of a test is the smallest significance level for which the null hypothesis is rejected. So this corresponds to the largest value of c_α for which the test will reject H_0. Clearly, we must take $c_\alpha = 28$ since is $c_\alpha > 28$ we don't reject. The type I error of this test (i.e., the p-value) is then

$$
\begin{aligned}
\textbf{p-value} &= P_{\lambda=0.25}(Y \geq 28) = P_{\lambda=0.25}(Y \geq 28) = 1 - P_{\lambda=0.25}(Y < 28) \\
&= 1 - \left(1 - e^{-28\lambda} \sum_{k=0}^{n-1} \frac{(28\lambda)^k}{k!}\right) = e^{-28\lambda}\left(1 + 28\lambda + \frac{(28\lambda)^2}{2}\right) \\
&= e^{-7}\left(1 + 7 + 7^2/2\right) \approx 0.02964.
\end{aligned}
$$

Since the p-value is smaller than 0.05 we would reject the null hypothesis at significantly level $\alpha = 0.05$.

Prob.IV: Consider the following situation. You are spending some well deserved holidays on the beach, in a country with consistently good weather. While relaxing on the sand you notice that the number of swimmers in the water seems heavily influenced by the water temperature. This prompts the question: can you use the number of swimmers as a "thermometer"?

Over the course of two weeks you count the number of swimmers entering the water between 10:30 and 11:00 in a pre-selected region of the beach. In addition, you take note of the water temperature (as reported by a local meteorological site). Suppose that the data below is what you collected.

Day	Number of swimmers	Water temperature (°C)
1	57	19.0
2	88	19.5
3	89	23.5
4	73	17.5
5	91	21.0
6	100	20.0
7	70	21.5
8	109	22.0
9	101	25.0
10	91	20.5
11	79	22.0
12	96	23.5
13	82	22.5
14	101	23.5

A simple linear regression analysis is to be conducted where the water temperature is taken as the response variable and the number of swimmers as the predictor. **Have a look at Appendix A before you start solving this problem.**

3 pts (a) Write the assumed **model equation** for the relation between the number of swimmers and the water temperature using α to denote the intercept and β the slope. Suppose that you fit the model and get the plot in Figure 2 for the resulting residuals. Is it reasonable to assume normally distributed errors?

Solution: Let y_i denote the temperature of the water on the i-th day, and x_i denote the number of swimmers. In the regression model we assume that

$$y_i = \alpha + \beta x_i + \epsilon_i,$$

where $\alpha, \beta \in \mathbb{R}$ are unknown parameters and ϵ_i are zero mean independent random errors. From Figure 2 we see that most points lie relatively close to a straight line in the normal QQ plot of the residuals. Therefore the normality assumption is somewhat reasonable.

8 pts (b) Estimate the model parameters α and β from the data, as well as σ^2, the variance of the noise.

Solution: We have that

$$S_{xx} = \sum_{i=1}^{n} x_i^2 - n(\bar{x})^2 = 2631.214,$$
$$S_{yy} = \sum_{i=1}^{n} y_i^2 - n(\bar{y})^2 = 55.5,$$
$$S_{xy} = \sum_{i=1}^{n} x_i y_i - n\bar{y}\bar{x} = 204.5,$$
$$\hat{\beta} = S_{xy}/S_{xx} = 0.07772077,$$
$$\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x} = 14.68833,$$
$$\hat{\sigma}^2 = S_{yy}/n - \hat{\beta}^2 S_{xx}/n = 2.829007.$$

3 pts (c) Suppose that after a quick trip to the country side you returned to the same beach and decided to see if your model was indeed useful in predicting the water temperature. Between 10:30 and 11:00 you counted 93 swimmers. Give an estimate for the water temperature (according to your model).

Solution: Let $x_0 = 93$. A point estimate for the water temperature is

$$\hat{\mu} = \hat{\alpha} + 93 \times \hat{\beta} = 14.68833 + 93 \times 0.07772 = 21.92 \textbf{ degrees}.$$

9 pts (d) Your friends were impressed by your model, but a bit doubtful about the quality of your estimate in (c) and wanted to have a better idea of the errors that are involved. Use your knowledge of regression models to give a two sided **prediction interval** for the water temperature in the scenario in (c) (use $\alpha = 0.05$).

Solution: We need to compute a two-sided prediction interval for the temperature. The end-points of this prediction interval are

$$\hat{\mu} \pm t_{1-\alpha/2;14-2} \sqrt{\hat{\sigma}^2 \left(1 + \frac{1}{n} + \frac{(x_0 - \bar{x})^2}{S_{xx}}\right)}.$$

Note that $t_{0.975;12} = 2.179$ (looked up in the 0.025 column of the table of t-probabilites.) Plugging in everything, we conclude the prediction interval is $[18.10346, 25.72926]$.

A Beach Data

From the dataset we get $\bar{x} = 87.64286$, $\bar{y} = 21.5$, $\sum_{i=1}^{n} x_i^2 = 110169$, $\sum_{i=1}^{n} y_i^2 = 6527$, $\sum_{i=1}^{n} x_i y_i = 26585$.

Below follows a normal QQ-plot of the residuals of the model.

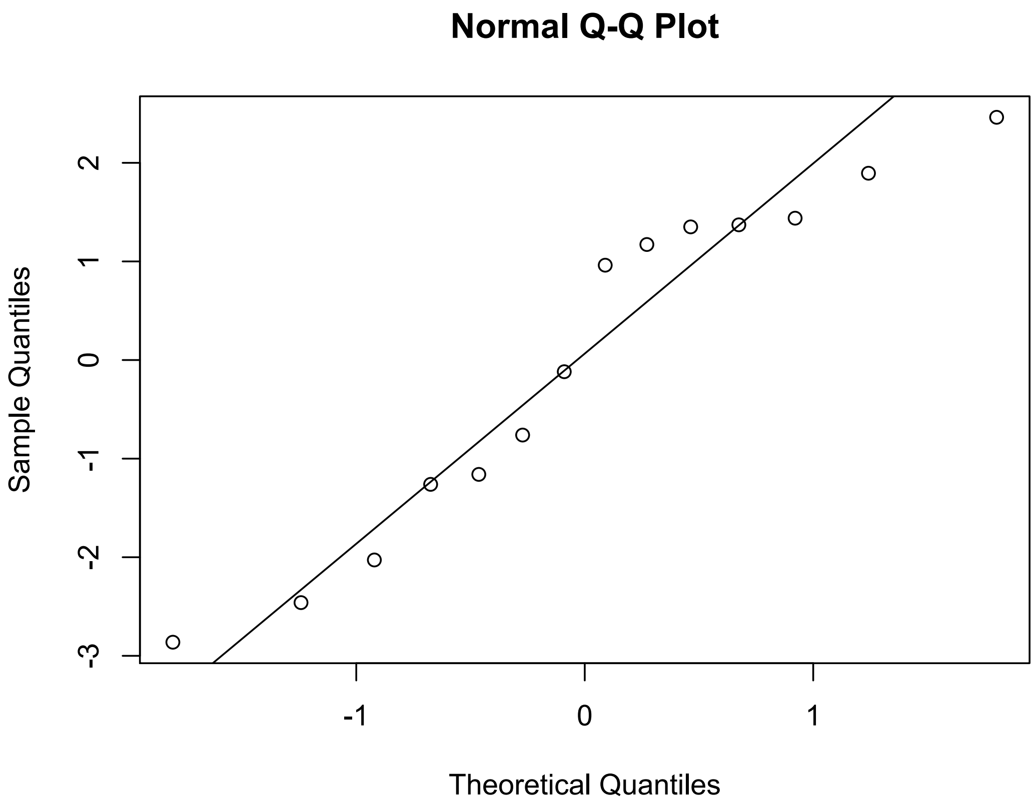

Figure 2: Normal QQ plot of the residuals of the regression model.

B Quantiles and cumulative probability tables

Below you can find a table of probabilities for the t-distribution and for the standard normal distribution, respectively. Please read the caption of the tables carefully.

$\downarrow \nu/\alpha \rightarrow$	0.3	0.2	0.15	0.1	0.05	0.025	0.02	0.01	0.005	0.0025	0.001
1	0.727	1.376	1.963	3.078	6.314	12.71	15.90	31.82	63.66	127.3	318.3
2	0.617	1.061	1.386	1.886	2.920	4.303	4.849	6.965	9.925	14.10	22.33
3	0.584	0.978	1.250	1.638	2.353	3.182	3.482	4.541	5.841	7.453	10.215
4	0.569	0.941	1.190	1.533	2.132	**2.776**	2.999	3.747	4.604	5.598	7.173
5	0.559	0.920	1.156	1.476	2.015	2.571	2.757	3.365	4.032	4.773	5.893
6	0.553	0.906	1.134	1.440	1.943	2.447	2.612	3.143	3.707	4.317	5.208
7	0.549	0.896	1.119	1.415	1.895	2.365	2.517	2.998	3.499	4.029	4.785
8	0.546	0.889	1.108	1.397	1.860	2.306	2.449	2.896	3.355	3.833	4.501
9	0.543	0.883	1.100	1.383	1.833	2.262	2.398	2.821	3.250	3.690	4.297
10	0.542	0.879	1.093	1.372	1.812	2.228	2.359	2.764	3.169	3.581	4.144
11	0.540	0.876	1.088	1.363	1.796	2.201	2.328	2.718	3.106	3.497	4.025
12	0.539	0.873	1.083	1.356	1.782	2.179	2.303	2.681	3.055	3.428	3.930
13	0.538	0.870	1.079	1.350	1.771	2.160	2.282	2.650	3.012	3.372	3.852
14	0.537	0.868	1.076	1.345	1.761	2.145	2.264	2.624	2.977	3.326	3.787
15	0.536	0.866	1.074	1.341	1.753	2.131	2.249	2.602	2.947	3.286	3.733
16	0.535	0.865	1.071	1.337	1.746	2.120	2.235	2.583	2.921	3.252	3.686
17	0.534	0.863	1.069	1.333	1.740	2.110	2.224	2.567	2.898	3.222	3.646
18	0.534	0.862	1.067	1.330	1.734	2.101	2.214	2.552	2.878	3.197	3.610
19	0.533	0.861	1.066	1.328	1.729	2.093	2.205	2.539	2.861	3.174	3.579
20	0.533	0.860	1.064	1.325	1.725	2.086	2.197	2.528	2.845	3.153	3.552
21	0.532	0.859	1.063	1.323	1.721	2.080	2.189	2.518	2.831	3.135	3.527
22	0.532	0.858	1.061	1.321	1.717	2.074	2.183	2.508	2.819	3.119	3.505
23	0.532	0.858	1.060	1.319	1.714	2.069	2.177	2.500	2.807	3.104	3.485
24	0.531	0.857	1.059	1.318	1.711	2.064	2.172	2.492	2.797	3.091	3.467
25	0.531	0.856	1.058	1.316	1.708	2.060	2.167	2.485	2.787	3.078	3.450
26	0.531	0.856	1.058	1.315	1.706	2.056	2.162	2.479	2.779	3.067	3.435
27	0.531	0.855	1.057	1.314	1.703	2.052	2.158	2.473	2.771	3.057	3.421
28	0.530	0.855	1.056	1.313	1.701	2.048	2.154	2.467	2.763	3.047	3.408
29	0.530	0.854	1.055	1.311	1.699	2.045	2.150	2.462	2.756	3.038	3.396
∞	0.524	0.842	1.036	1.282	1.645	1.960	2.054	2.326	2.576	2.807	3.090

Figure 3: Right quantiles of the student t-distribution with ν degrees of freedom. Example: if X is a student t distributed random variable, with $\nu = 4$ degrees of freedom then $P(X \geq 2.776) = 0.025$. **The entries of the table are therefore $t_{\nu,1-\alpha}$.**

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	**0.9115**	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Figure 4: Cumulative probabilities of the standard normal distribution. The rows correspond to the number rounded down to the closest decimal, and columns represent the second decimal. Example: if Z is a standard normal random variable then $P(Z \leq 1.35) = 0.9115$. You look this number up in the row corresponding to 1.3, and in the 6th column (corresponding to 0.05.) Note that for $z < 0$ we have $P(Z \leq z) = 1 - P(Z \leq -z)$. So for example $P(Z \leq -1.35) = 1 - 0.9115 = 0.0885$.