X_400004 - Statistics
Midterm Resit

9 February 2021

Instructions:

- The exam is to be solved **individually**.
- Please **write clearly and in an organised way**: we can't grade illegible answers.
- Please change pages when starting a new question.
- This is an exam on a mathematical subject, so support your answers with **computations**, rather than words, whenever possible.
- You should report **all relevant computations** and **justify** non-trivial steps.
- This is a **closed book exam**; you are only allowed to have one A4 sheet with **handwritten** notes with you.
- You may use a calculator.
- There are 6 pages in the exam questionnaire (including this one) and you have two hours (120 minutes) to complete the exam.
- Students entitled to extra time have an extra 20 minutes.
- The exam consists of 12 questions spread throughout 3 problems.
- The number of points per question is indicated next to it for a total of 100 points.
- Your final grade is $\max(1, \text{score}/10)$, where "score" is the number you points you get.
- The problems are not necessarily ordered in term of difficulty. I recommend that you quickly read through all problems first, then do the problems in whatever order suits you best.
- Remember to **identify** at least the first page of your answer sheets with you name, course name, and student number.

After completing your exam, digitalise your answer sheets (with the correct order and orientation) and submit them as **a single PDF** on Canvas for grading. **You have 10 minutes following the end of the exam to do this** after which any submission will be marked as late. These instructions do not replace the VU's *Protocol of online examination for 2020-2021* that you can find on Canvas.

Prob.I: In the study of wireless communication networks one often comes across a process that is well modeled by samples from continuous random variables with density

$$f(x) = \begin{cases} \frac{1}{\sqrt{\lambda}} e^{-\frac{x}{\sqrt{\lambda}}} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases},$$

where $\lambda > 0$ is a parameter we would like to estimate. Suppose you observe two **independent** samples X_1 and X_2 from the above distribution and consider the following two possible estimators of λ:

$$\hat{\lambda} = \frac{X_1^2 + X_2^2}{4} \qquad \text{and} \qquad \tilde{\lambda} = \frac{X_1^2 + X_2^2 + 2X_1X_2}{6}.$$

You can use the fact that, if X is a random variable with the density above then

$$\mathbb{E}(X) = \sqrt{\lambda}, \qquad \mathbb{E}(X^2) = 2\lambda, \qquad \mathbb{E}(X^3) = 6\lambda\sqrt{\lambda}, \qquad \mathbb{E}(X^4) = 24\lambda^2.$$

14 pts (a) Compute the bias of the two estimators. Is there an estimator that is biased?

12 pts (b) Compute the variance of $\hat{\lambda}$.

6 pts (c) Compute the MSE of estimator $\hat{\lambda}$.

6 pts (d) It can be shown that the MSE of $\tilde{\lambda}$ is given by $\frac{7}{3}\lambda^2$. Given this and your answer to (c) which of the two estimators would you prefer? Justify your answer.

Prob.II: Suppose that X is a discrete random variable that with the following distribution

$$\mathbb{P}(X=0)=\frac{2}{3}\theta,\quad \mathbb{P}(X=1)=\frac{1}{3}\theta,\quad \mathbb{P}(X=2)=\frac{2}{3}(1-\theta),\quad \mathbb{P}(X=3)=\frac{1}{3}(1-\theta),$$

where $0 \leq \theta \leq 1$ is an unknown parameter. The following 10 independent observations were taken from such a distribution: $\{3,0,2,1,3,2,1,0,2,1\}$.

7 pts (a) Find the method of moments estimator of θ. What is the method of moments estimate?

7 pts (b) What is the maximum likelihood estimate of θ?

7 pts (c) If the prior distribution on θ is $U[0,1]$, what is the posterior density?

4 pts (d) Sketch the posterior. What is the mode of the posterior?

Prob.III: Suppose that you are managing an email mailing list for a website. Let $X_1, \ldots, X_n$ be a random sample from the Bernoulli distribution with unknown parameter $p \in [0, 1]$. Each observation X_i corresponds to a different person that got the mailing list email. Suppose that $X_i = 1$ if the person opened the email, and $X_i = 0$, otherwise. You are interested in the value of p, the probability that someone who gets the mailing list actually opens it. (The other parameter in the model, n, is not specified but you always know what it is.)

7 pts (a) Describe a near-pivot T for p. Justify how you arrived at T, and don't forget to mention the (approximate) distribution of the pivot.

12 pts (b) Use the near-pivot from the previous question to derive a confidence interval of level 0.95 for p. Don't just present the final result: show how you go from (near-)pivot to confidence interval.

7 pts (c) Suppose now that the last issue of the mailing list was sent to $n = 27$ subscribers and that, out of these, only 2 opened the email. Based on the confidence interval for p that you found, what would be your most optimistic guess as to how large p is? (There are tables at the end of the exam questionnaire in Appendix A that you may need to consult to answer this question.)

11 pts (d) Suppose that you are not happy with the large amount of uncertainty in the confidence interval that you got – you find the confidence interval too wide. In class, we saw that increasing the sample size n typically reduces uncertainty. How many emails would you have to send to be fairly confident that the estimate $\bar{X}$ of p is not off by more that 0.005? (You can still use $\bar{X}$ as your best guess for p.)

A Quantiles and cumulative probability tables

Below you can find a table of probabilities for the t-distribution and for the standard normal distribution, respectively. Please read the caption of the tables carefully.

$\downarrow \nu/\alpha \rightarrow$	0.3	0.2	0.15	0.1	0.05	0.025	0.02	0.01	0.005	0.0025	0.001
1	0.727	1.376	1.963	3.078	6.314	12.71	15.90	31.82	63.66	127.3	318.3
2	0.617	1.061	1.386	1.886	2.920	4.303	4.849	6.965	9.925	14.10	22.33
3	0.584	0.978	1.250	1.638	2.353	3.182	3.482	4.541	5.841	7.453	10.215
4	0.569	0.941	1.190	1.533	2.132	**2.776**	2.999	3.747	4.604	5.598	7.173
5	0.559	0.920	1.156	1.476	2.015	2.571	2.757	3.365	4.032	4.773	5.893
6	0.553	0.906	1.134	1.440	1.943	2.447	2.612	3.143	3.707	4.317	5.208
7	0.549	0.896	1.119	1.415	1.895	2.365	2.517	2.998	3.499	4.029	4.785
8	0.546	0.889	1.108	1.397	1.860	2.306	2.449	2.896	3.355	3.833	4.501
9	0.543	0.883	1.100	1.383	1.833	2.262	2.398	2.821	3.250	3.690	4.297
10	0.542	0.879	1.093	1.372	1.812	2.228	2.359	2.764	3.169	3.581	4.144
11	0.540	0.876	1.088	1.363	1.796	2.201	2.328	2.718	3.106	3.497	4.025
12	0.539	0.873	1.083	1.356	1.782	2.179	2.303	2.681	3.055	3.428	3.930
13	0.538	0.870	1.079	1.350	1.771	2.160	2.282	2.650	3.012	3.372	3.852
14	0.537	0.868	1.076	1.345	1.761	2.145	2.264	2.624	2.977	3.326	3.787
15	0.536	0.866	1.074	1.341	1.753	2.131	2.249	2.602	2.947	3.286	3.733
16	0.535	0.865	1.071	1.337	1.746	2.120	2.235	2.583	2.921	3.252	3.686
17	0.534	0.863	1.069	1.333	1.740	2.110	2.224	2.567	2.898	3.222	3.646
18	0.534	0.862	1.067	1.330	1.734	2.101	2.214	2.552	2.878	3.197	3.610
19	0.533	0.861	1.066	1.328	1.729	2.093	2.205	2.539	2.861	3.174	3.579
20	0.533	0.860	1.064	1.325	1.725	2.086	2.197	2.528	2.845	3.153	3.552
21	0.532	0.859	1.063	1.323	1.721	2.080	2.189	2.518	2.831	3.135	3.527
22	0.532	0.858	1.061	1.321	1.717	2.074	2.183	2.508	2.819	3.119	3.505
23	0.532	0.858	1.060	1.319	1.714	2.069	2.177	2.500	2.807	3.104	3.485
24	0.531	0.857	1.059	1.318	1.711	2.064	2.172	2.492	2.797	3.091	3.467
25	0.531	0.856	1.058	1.316	1.708	2.060	2.167	2.485	2.787	3.078	3.450
26	0.531	0.856	1.058	1.315	1.706	2.056	2.162	2.479	2.779	3.067	3.435
27	0.531	0.855	1.057	1.314	1.703	2.052	2.158	2.473	2.771	3.057	3.421
28	0.530	0.855	1.056	1.313	1.701	2.048	2.154	2.467	2.763	3.047	3.408
29	0.530	0.854	1.055	1.311	1.699	2.045	2.150	2.462	2.756	3.038	3.396
∞	0.524	0.842	1.036	1.282	1.645	1.960	2.054	2.326	2.576	2.807	3.090

Figure 1: Right quantiles of the student t-distribution with ν degrees of freedom. Example: if X is a student t distributed random variable, with $\nu = 4$ degrees of freedom then $P(X \geq 2.776) = 0.025$. **The entries of the table are therefore $t_{\nu,1-\alpha}$.**

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	**0.9115**	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Figure 2: Cumulative probabilities of the standard normal distribution. The rows correspond to the number rounded down to the closest decimal, and columns represent the second decimal. Example: if Z is a standard normal random variable then $P(Z \leq 1.35) = 0.9115$. You look this number up in the row corresponding to 1.3, and in the 6th column (corresponding to 0.05.) Note that for $z < 0$ we have $P(Z \leq z) = 1 - P(Z \leq -z)$. So for example $P(Z \leq -1.35) = 1 - 0.9115 = 0.0885$.