Department of Mathematics	short answers midterm exam Statistics
VU	October 25, 2019

1. (a) Since p_θ is symmetric, we have $\mathbb{E}_\theta X_1 = 0$. We have

$$\mathbb{E}_\theta X_1^2 = \int_{\mathbb{R}} x^2 p_\theta(x)\,dx = \frac{1}{\theta^2}\int_0^\infty x^2 e^{-x}\,dx = \frac{2}{\theta^2}.$$

Hence the moment estimator is the solution to $2/\theta^2 = \overline{X^2}$, which gives the estimator $\sqrt{2/\overline{X^2}}$.

(b) The likelihood is

$$L(\theta; X_1,\ldots,X_n) = \prod_{i=1}^n p_\theta(X_i) = \frac{1}{2^n}\theta^n e^{-\theta\sum_{i=1}^n |X_i|}$$

Hence the log-likelihood is

$$\ell(\theta; X_1,\ldots,X_n) = \log\frac{1}{2^n} + n\log\theta - \theta\sum_{i=1}^n |X_i|$$

and the score function is

$$\dot{\ell}(\theta; X_1,\ldots,X_n) = \frac{n}{\theta} - \sum_{i=1}^n |X_i|.$$

The MLE is obtained by setting this equal to 0 and checking for a maximum. Hence, the MLE is $n/\sum_{i=1}^n |X_i|$.

(c) The density $p_{\bar{\Theta}\,|\,X_1,\ldots,X_n}$ of the posterior is proportional to the density of the prior times the likelihood. Hence,

$$p_{\bar{\Theta}\,|\,X_1,\ldots,X_n}(\theta) \propto \theta^{\alpha+n-1}e^{-\theta(\beta+\sum_{i=1}^n |X_i|)}.$$

This is up to a constant equal to the density of a gamma distribution with parameters $\alpha+n$ and $\beta+\sum_{i=1}^n |X_i|$. Hence, the latter is the posterior distribution.

(d) The Bayes estimator is the posterior mean. Hence, it is given by $(\alpha+n)/(\beta+\sum_{i=1}^n |X_i|)$.

2. (a) The Fisher information in a single observation is given by

$$i_\lambda = \mathbb{V}\mathrm{ar}_\lambda \frac{\partial}{\partial\lambda}\log p_\lambda(X_1).$$

Working this out gives $i_\lambda = 1/\lambda^2$. Hence, the Fisher information in the whole observation is $I_\lambda = n i_\lambda = n/\lambda^2$.

(b) The lower bound is $1/I_\lambda$, which equals λ^2/n.

(c*) The MLE is $\hat{\lambda}_n = \overline{X}$ (1 point). By the CLT,

$$\sqrt{n}(\bar{X} - \mathbb{E}_\lambda X_1) \xrightarrow{\text{d}} N(0, \mathbb{V}\text{ar}_\lambda X_1).$$

This prove the statement, since $\mathbb{E}_\lambda X_1 = \lambda$ and $\mathbb{V}\text{ar}_\lambda X_1 = \lambda^2$.

(d*) Either from exact computations for fixed n, or from the asymptotic considerations of part (c), we see that the MLE is (approximately) unbiased and has the minimal variance obtained in part (b).

3. (a) $X \sim \text{Bin}(200, p)$.

(b) $H_0 : p \le 1/2$; $H_1 : p > 1/2$.

(c)
- *Test statistic*: X.
- *Form of the critical region*: large values of X indicate that H_1 is true. Hence take a test of the form "reject H_0 if $X \ge c$", for an appropriate $c \in \{0, 1, \ldots, n\}$.
- *Final critical region*: Want a test of level $\alpha = 0.05$, i.e. want

$$\sup_{p \le 1/2} \mathbb{P}_p(X \ge c) \le 0.05.$$

The probability is increasing in p, hence the requirement is equivalent to $\mathbb{P}_{1/2}(X \ge c) \le 0.05$. This is equivalent to $\mathbb{P}_{1/2}(X \le c - 1) \ge 0.95$. From the "table" that is given we then read off that we should take $c - 1 \ge 112$. Since we want to have c as small as possible, we take $c = 113$.

(d) 111 is not in the critical region, so we can not conclude from the data that H_1 is true.

4. (a) $H_0 : \mu \le 0$; $H_1 : \mu > 0$. The standard test statistic for this test is $T = \sqrt{n}\bar{X}$ ($\bar{X}$ is also correct).

(b) We should reject H_0 for large values of T (or $\bar{X}$) (so this is a right tailed test). The observed value of T is $t = \sqrt{n}\bar{x}$. Hence, the p-value is given by

$$\sup_{\mu \le 0} \mathbb{P}_\mu(T \ge \sqrt{n}\bar{x}).$$

The probability is increasing in μ, hence this equals

$$\mathbb{P}_0(T \ge \sqrt{n}\bar{x}).$$

If $\mu = 0$, then T is standard normal. Hence, this further equals

$$1 - \Phi(\sqrt{n}\bar{x}).$$