Department of Mathematics	practice midterm exam Statistics
VU	

- **You are not allowed to use calculators, phones, laptops, or other tools.**

Good luck!

1. Let $X_1, \ldots, X_n$ be i.i.d. random variables with density p_θ given by

$$p_\theta(x) = \theta^2 x e^{-\theta x}, \quad x \geq 0,$$

where $\theta > 0$ is an unknown parameter.

   (a) Determine the method of moments estimator for θ.

   (b) Determine the maximum likelihood estimator for θ.

2. Let $X_1, \ldots, X_n$ be i.i.d. random variables with density p_θ given by

$$p_\theta(x) = \frac{\sqrt{\theta}}{\sqrt{2\pi}} e^{-\frac{1}{2}\theta x^2}, \quad x \in \mathbb{R},$$

where $\theta > 0$ is an unknown parameter. We are going to estimate this parameter Bayesian style, using as prior distribution a gamma distribution with fixed parameters $r, \lambda > 0$. This has density

$$\pi(\theta) = \frac{\lambda^r}{\Gamma(r)} \theta^{r-1} e^{-\lambda\theta}, \quad \theta > 0,$$

where Γ is the gamma function (the precise form of Γ is irrelevant), expectation r/λ and variance r/λ^2.

   (a) Determine the posterior distribution. Which known distribution is this?

   (b) Determine the Bayes estimator for θ.

3. Let $X_1, \ldots, X_n$ be i.i.d. random variables with density

$$p_\lambda(x) = \lambda e^{-\lambda x}, \quad x > 0,$$

where $\lambda > 0$ is an unknown parameter.

   (a) Determine the Fisher information for λ in the whole vector $(X_1, \ldots, X_n)$.

   (b) Determine the Cramér-Rao lower bound for the variance of an unbiased estimator for $1/\lambda$.

(c) Show that the maximum likelihood estimator (MLE) for $1/\lambda$ is approximately normally distributed for large n.

(d) Show that for large n, the MLE for $1/\lambda$ is approximately unbiased and has minimal variance.

4. An opinion poller wants to investigate which fraction p of the Dutch people think Virgil van Dijk is the best soccer player in Europe. She asks 900 people. Let X be the number of people that answered that Virgil is the best player.

   (a) If we assume that the people have been chosen completely at random and independently, what is a reasonable model for the distribution of X?

   (b) The poller wants to show that more than 50% of the people think that Van Dijk is the best player. Formulate this as a testing problem.

   (c) Suppose that of the 900 people, x say that Van Dijk is the best. For this situation, give a (appropximate) p-value for the testing problem of part (b). (You may use the usual approximation.)

5. Let $X_1, \ldots, X_n$ be i.i.d. and $N(\mu, \sigma^2)$-distributed, with unknown parameter $\mu \in \mathbb{R}$ and known $\sigma^2 > 0$.

   (a) We want to show that $\mu > 1$. Formulate this as a testing problem and give an appropriate test statistic.

   (b) Give a derivation of a test of level $\alpha \in (0, 1)$ for this problem.