Exercise 1 [26 points (midterm retake); 18 points (full retake)]
Let $X_1, \dots, X_n$ be independent random variables with density

$$p_\theta(x) = \begin{cases} \frac{1}{\theta} e^{1-\frac{x}{\theta}}, & x \geq \theta, \\ 0, & x < \theta, \end{cases}$$

where $\theta > 0$ is an unknown parameter.

(a) [9 points] Give the moment estimator $\widehat{\theta}_{\text{MM}}$ and the maximum likelihood estimator $\widehat{\theta}_{\text{ML}}$ of the parameter θ.
Hint 1: To derive $\widehat{\theta}_{\text{MM}}$, use without derivation the fact that $X_i = Y_i + \theta$, where the Y_i's are independent and follow the Exponential$(1/\theta)$ distribution (see Appendix 1).
Hint 2: To derive $\widehat{\theta}_{\text{ML}}$, check where the (log-)likelihood function is increasing and keep in mind that $X_{(1)} \leq \overline{X}$.

By *Hint 1* and Appendix 1, $\mathbb{E}X_1 = \mathbb{E}Y_1 + \theta = \frac{1}{1/\theta} + \theta = 2\theta$. We have $\theta = \frac{1}{2}\mathbb{E}X_1$ and the moment estimator is given by $\widehat{\theta}_{\text{MM}} = \frac{1}{2}\overline{X}$.
The log-likelihood function is

$$\begin{aligned} \ln L(\theta) &= \ln \prod_{i=1}^n p_\theta(X_i) = \sum_{i=1}^n \ln p_\theta(X_i) = \sum_{i=1}^n \ln \left(\frac{1}{\theta} e^{1-\frac{X_i}{\theta}} \right) \\ &= \sum_{i=1}^n (-\ln\theta + 1 - \frac{X_i}{\theta}) = n \underbrace{-n\ln\theta - \frac{\sum_{i=1}^n X_i}{\theta}}_{=:\, f(\theta)}. \end{aligned}$$

The data implies an additional restriction on the possible values of the parameter θ: since $X_i \in [\theta, \infty)$ for all i, we have $\theta \leq X_{(1)}$. The maximum likelihood estimator is given by

$$\widehat{\theta}_{\text{ML}} = \text{argmax}_{\theta \leq X_{(1)}} L(\theta) = \text{argmax}_{\theta \leq X_{(1)}} \ln L(\theta) = \text{argmax}_{\theta \leq X_{(1)}} f(\theta).$$

We have

$$f'(\theta) = -\frac{n}{\theta} + \frac{\sum_{i=1}^n X_i}{\theta^2} = \frac{n}{\theta^2}(-\theta + \overline{X}) \begin{cases} > 0, & \theta < \overline{X}, \\ = 0, & \theta = \overline{X}, \\ < 0, & \theta > \overline{X}. \end{cases}$$

We have to maximize $f(\theta)$ over $\theta \leq X_{(1)}$. By the above, $f(\theta)$ is increasing on $\theta \leq \overline{X}$, and then *Hint 2* implies that $f(\theta)$ is increasing on $\theta \leq X_{(1)}$. That is,

$$\widehat{\theta}_{\text{ML}} = \text{argmax}_{\theta \leq X_{(1)}} f(\theta) = X_{(1)}.$$

(b) [8 points (midterm retake only)] Transform the estimators $\widehat{\theta}_{\text{MM}}$ and $\widehat{\theta}_{\text{ML}}$ that you found in part *(a)* into unbiased estimators $\widehat{\theta}_1$ and $\widehat{\theta}_2$ of the parameter θ. (If $\widehat{\theta}_{\text{MM}}$ is already unbiased, just put $\widehat{\theta}_1 = \widehat{\theta}_{\text{MM}}$. If $\widehat{\theta}_{\text{ML}}$ is already unbiased, just put $\widehat{\theta}_2 = \widehat{\theta}_{\text{ML}}$.)
Hint 3: In addition to *Hint 1*, note that $X_{(1)} = Y_{(1)} + \theta$. To figure the distribution of $Y_{(1)}$, use without derivation the fact that, for independent random variables $E_i \sim \text{Exponential}(\lambda_i)$, $i = 1, \dots, n$, we have $\min_{i=1,\dots,n} E_i \sim \text{Exponential}(\sum_{i=1}^n \lambda_i)$.

We have

$$\mathbb{E}\widehat{\theta}_{\text{MM}} = \frac{1}{2}\mathbb{E}\overline{X} = \frac{1}{2}\mathbb{E}X_1 \overset{(a)}{=} \frac{1}{2} \cdot 2\theta = \theta.$$

That is, $\widehat{\theta}_{\text{MM}}$ is an unbiased estimator for θ and we put $\widehat{\theta}_1 = \frac{1}{2}\overline{X}$.
By *Hints 1* and *3*, $Y_{(1)} = \min_{i=1,\dots,n} Y_i \sim \text{Exponential}(n/\theta)$ and $X_{(1)} = Y_{(1)} + \theta$. Then, by Appendix 1,

$$\mathbb{E}\widehat{\theta}_{\text{ML}} = \mathbb{E}X_{(1)} = \mathbb{E}Y_{(1)} + \theta = \frac{1}{n/\theta} + \theta = \theta\frac{n+1}{n}.$$

That is, $\widehat{\theta}_{\text{ML}}$ is biased. It can be transformed into an unbiased estimator by scaling: put

$$\widehat{\theta}_2 = \widehat{\theta}_{\text{ML}}\frac{n}{n+1} = X_{(1)}\frac{n}{n+1},$$

then

$$\mathbb{E}\widehat{\theta}_2 = \frac{n}{n+1}\mathbb{E}\widehat{\theta}_{\text{ML}} = \frac{n}{n+1}\theta\frac{n+1}{n} = \theta.$$

That is, $\widehat{\theta}_2 = X_{(n)}\frac{n}{n+1}$ is an unbiased estimator for θ.

(c) [9 points] Which of the unbiased estimators $\widehat{\theta}_1$ and $\widehat{\theta}_2$ that you found in part *(b)* is better for large data sets? (That is, as $n \to \infty$. You do not have to specify for which n exactly one estimator is better than the other). Use *Hints 1, 3* and

Hint 4: If you have no answer to part *(b)*, compare $\widehat{\theta}_1 = \frac{1}{2}\overline{X}$ and $\widehat{\theta}_2 = \frac{n}{n+1}X_{(1)}$ under the assumption that these are unbiased estimators for θ.

Since $\widehat{\theta}_1$ is unbiased, we have

$$\begin{aligned}\text{MSE}(\widehat{\theta}_1) &= \text{Var}\widehat{\theta}_1 = \frac{1}{4}\text{Var}\overline{X} = \frac{1}{4}\cdot\frac{\text{Var}X_1}{n} \overset{\textit{Hint 1}}{=} \frac{1}{4n}\text{Var}(Y_1+\theta)\\ &= \frac{1}{4n}\text{Var}(\underbrace{Y_1}_{\sim\text{Exponential}(1/\theta)}) \overset{\text{Appendix 1}}{=} \frac{1}{4n}\frac{1}{(1/\theta)^2} = \frac{\theta^2}{4n}.\end{aligned}$$

Since $\widehat{\theta}_2$ is unbiased as well,

$$\text{MSE}(\widehat{\theta}_2) = \text{Var}\widehat{\theta}_2 = \frac{n^2}{(n+1)^2}\text{Var}X_{(1)} \overset{\textit{Hint 1}}{=} \frac{n^2}{(n+1)^2}\text{Var}(Y_{(1)}+\theta) = \frac{n^2}{(n+1)^2}\text{Var}Y_{(1)},$$

where $Y_{(1)} \sim \text{Exponential}(n/\theta)$ by *Hint 3*, and hence, by Appendix 1,

$$\text{MSE}(\widehat{\theta}_2) = \frac{n^2}{(n+1)^2}\text{Var}Y_{(1)} = \frac{n^2}{(n+1)^2}\frac{1}{(n/\theta)^2} = \frac{\theta^2}{(n+1)^2}.$$

As $n \to \infty$, we have $\text{MSE}(\widehat{\theta}_2) < \text{MSE}(\widehat{\theta}_1)$ because $(n+1)^2$ grows faster than $4n$. Hence, for large data sets, the estimator $\widehat{\theta}_2$ is better than $\widehat{\theta}_1$.

Exercise 2 [8 points]

Let $X_1, \dots, X_n$ be independent random variables with the density

$$p_\theta(x) = \begin{cases}\theta x^{\theta-1}, & 0 \le x \le 1,\\ 0, & x < 0,\ x > 1,\end{cases}$$

where $\theta > 0$ is an unknown parameter. Give the Bayes estimator for θ under the Exponential(1) prior belief about θ, i.e. the prior density is given by

$$\pi(\theta) = \begin{cases}e^{-\theta}, & \theta > 0,\\ 0, & \theta \le 0.\end{cases}$$

Hint 5: The posterior parameter density is proportional to the product of the prior parameter density and the likelihood function. The posterior distribution is a common distribution, see Appendix 1.

The posterior density is proportional to

$$p_{\overline{\Theta}|X_1,\ldots,X_n}(\theta) \propto \pi(\theta)\prod_{i=1}^{n} p_\theta(X_i) = e^{-\theta}\theta^n \prod_{i=1}^{n}(X_i^{\theta-1}) = e^{-\theta}\theta^n \left(\prod_{i=1}^{n} X_i\right)^{\theta} \left(\prod_{i=1}^{n} X_i\right)^{-1}$$

$$\propto e^{-\theta}\theta^n \left(\prod_{i=1}^{n} X_i\right)^{\theta} = \theta^n e^{\theta[-1+\ln(\prod_{i=1}^{n} X_i)]}, \quad \theta > 0.$$

We can match the posterior density to a Gamma density (see Appendix 1): $n = \alpha - 1$ and $-1 + \ln(\prod_{i=1}^{n} X_i) = -\lambda$. That is, $\overline{\Theta}|X_1, \ldots, X_n \sim \text{Gamma}(\alpha, \lambda)$ with

$$\alpha = n + 1, \quad \lambda = 1 - \ln\left(\prod_{i=1}^{n} X_i\right),$$

and the Bayes estimator for θ is

$$\widehat{\theta}_{\text{B}} = \mathbb{E}[\overline{\Theta}|X_1, \ldots, X_n] \overset{\text{Appendix 1}}{=} \frac{\alpha}{\lambda} = \frac{n+1}{1 - \ln(\prod_{i=1}^{n} X_i)}.$$

Exercise 3 [16 points (midterm retake only)]

A principal at a certain school suspects that the students in his school are of above average intelligence. Average intelligence corresponds to an IQ score of 100. A random sample of 30 students have a mean IQ score of 112. Does this provide sufficient evidence to support the principal's belief? Assume that the IQ scores of the students of this particular school follow a normal distribution with unknown mean μ and known standard deviation 15.

(a) [2 points] Formulate an appropriate statistical model and a null and alternative hypotheses.

IQ scores of different students are independent random variables $X_1, \ldots, X_{30} \sim N(\mu, 15^2)$, where the parameter μ is unknown.
We have to test $H_0 : \mu \leq 100$ VS $H_1 : \mu > 100$.

(b) [7 points] Test the hypotheses from part *(a)* so that the maximum chance of a type-1 error is 5%. Report the test statistic, its distribution on the border of H_0, the critical region, and the conclusion.

The test statistic is $\dfrac{\overline{X} - 100}{15/\sqrt{30}}$.
If $\mu = 100$ (border of H_0), $\dfrac{\overline{X} - 100}{15/\sqrt{30}} \sim N(0, 1)$.
The hypotheses should be tested at the significance level $\alpha_0 = 0.05$. We test as follows:

$$\underbrace{\frac{\overline{X} - 100}{15/\sqrt{30}} > \Phi^{-1}(1 - \alpha_0)}_{\text{critical region}} \Rightarrow \text{ reject } H_0,$$

$$\frac{\overline{X} - 100}{15/\sqrt{30}} \leq \Phi^{-1}(1 - \alpha_0) \Rightarrow \text{ fail to reject } H_0.$$

We observe $\dfrac{\overline{X} - 100}{15/\sqrt{30}} = \dfrac{112 - 100}{15/\sqrt{30}} = 4.38$ and $\Phi^{-1}(1 - \alpha_0) = \Phi^{-1}(0.95) = 1.645$. That is, the data are in the critical region ($4.38 > 1.645$) and we reject H_0. There is enough evidence to support the principal's belief.

(c) [7 points] How many students should be included in the sample to ensure that, in case the true mean μ of the IQ scores of the school's students is 110, the power of your test from *(b)* is at least 95%?

If there are n students in the sample, we test as follows:

$$\underbrace{\frac{\overline{X}-100}{15/\sqrt{n}} > \Phi^{-1}(1-\alpha_0) = 1.645}_{\text{critical region}} \Rightarrow \text{ reject } H_0.$$

The power of this test when $\mu = 110$ is

$$\begin{aligned}\mathbb{P}_{\mu=110}\{\text{reject } H_0\} &= \mathbb{P}_{\mu=110}\{\frac{\overline{X}-100}{15/\sqrt{n}} \geq 1.645\} = \mathbb{P}_{\mu=110}\{\frac{\overline{X}-110}{15/\sqrt{n}} \geq 1.645 - \frac{10}{15/\sqrt{n}}\}\} \\ &= \mathbb{P}\{N(0,1) \geq 1.645 - \frac{2}{3}\sqrt{n}\}.\end{aligned}$$

We have the power $\mathbb{P}_{\mu=4100}\{\text{reject } H_0\} = \mathbb{P}\{N(0,1) \geq 1.645 - \frac{2}{3}\sqrt{n}\} \geq 0.95$ if and only if $1.645 - \frac{2}{3}\sqrt{n} \leq \Phi^{-1}(0.05) = -\Phi^{-1}(0.95) = -1.645$, which is if and only if $n \geq 25$.

Exercise 4 [12 points]

In a packing plant, a machine packs cartons with jars. It is supposed that a new machine will pack faster on average than the machine currently used. To test that hypothesis, the times it takes each machine to pack ten cartons are recorded: $X_1, \ldots, X_{10}$ are the times of the new machine, $Y_1, \ldots, Y_{10}$ are the times of the old machine. The summaries of the observed times are $\overline{X} = 42.74$, $S_X = 0.683$, $\overline{Y} = 43.23$, $S_X = 0.75$. Carry out a suitable test at significance level 0.05 to investigate whether the data provide sufficient evidence to conclude that, on average, the new machine packs faster. Report

(a) [3 points] the statistical model and the null and alternative hypotheses,

$X_1, \ldots, X_{10}$ i.i.d. $\sim N(\mu, \sigma^2)$, $Y_1, \ldots, Y_{10}$ i.i.d. $\sim N(\nu, \sigma^2)$, the two samples are independent and have the same variance σ^2.

The new machine packing faster means its packing time is lower (putting this in H_1). So we want to test $H_0 : \mu \geq \nu$ VS $H_1 : \mu < \nu$.

(b) [4 points] the test statistic and its distribution when the new and the old machines are equally fast,
Hint 6: If $X_1, \ldots, X_m \sim N(\mu, \sigma^2)$ and $Y_1, \ldots, Y_n \sim N(\nu, \tau^2)$, all independent, then $\frac{(\overline{X}-\overline{Y})-(\mu-\nu)}{\sqrt{\sigma^2/m+\tau^2/n}} \sim N(0,1)$ and is independent from $\frac{(m-1)S_X^2}{\sigma^2} + \frac{(n-1)S_Y^2}{\tau^2}$, which has a chi-square distribution.

We apply the hint to $\mu = \nu$ and $\sigma^2 = \tau^2$. We have $\underbrace{\frac{(m-1)S_X^2}{\sigma^2} + \frac{(n-1)S_Y^2}{\sigma^2}}_{\text{bottom}} \sim \chi^2_{m+n-2}$ and is

independent from $\underbrace{\frac{\overline{X}-\overline{Y}}{\sqrt{\sigma^2/m+\sigma^2/n}}}_{\text{top}} \sim N(0,1)$. Hence $\frac{\text{top}}{\sqrt{\text{bottom}/(m+n-2)}} \sim t_{m+n-2}$, where

$$\frac{\text{top}}{\sqrt{\text{bottom}/(m+n-2)}} = \frac{\overline{X}-\overline{Y}}{\sigma\sqrt{1/m+1/n}} \Big/ \sqrt{\frac{(m-1)S_X^2+(n-1)S_Y^2}{\sigma^2}\cdot\frac{1}{m+n-2}}$$
$$= \frac{\overline{X}-\overline{Y}}{\sqrt{1/m+1/n}} \Big/ \sqrt{\frac{(m-1)S_X^2+(n-1)S_Y^2}{m+n-2}}.$$

That is, the test statistic is $T = \frac{\overline{X}-\overline{Y}}{\sqrt{1/m+1/n}} \Big/ \sqrt{\frac{(m-1)S_X^2+(n-1)S_Y^2}{m+n-2}}$ and its distribution under $\mu = \nu$ is t_{m+n-2}.

(c) [5 points] the critical region and the conclusion. At significance level α, we test as follows:

$$\underbrace{T < -t_{m+n-2,1-\alpha}}_{\text{critical region}} \Rightarrow \text{ reject } H_0,$$
$$T \geq -t_{m+n-2,1-\alpha} \Rightarrow \text{ fail to reject } H_0.$$

We have $m = n = 10$, $\alpha = 0.05$,

$$T = \frac{42.74-43.23}{\sqrt{1/10+1/10}} \Big/ \sqrt{\frac{9\cdot 0.683^2+9\cdot 0.75^2}{18}} = -1.53 \quad \text{and} \quad -t_{m+n-2,1-\alpha} = -t_{18,0.95} = -1.73.$$

Since $T = -1.53 \not< -t_{m+n-2} = -1.73$, we fail to reject H_0. That is, the data do not provide enough evidence that the new machine packs faster.

Exercise 5 [15 points]

Let $X_1, \dots, X_{10}$ be independent random variables such that, for all i, $X_i^3 \sim \text{Exponential}(1/\theta)$.

(a) [5 points] Give the definition of a pivot. Construct a pivot using (without derivation) the following hints.
Hint 7: If a random variable $E \sim \text{Exponential}(\lambda)$, then $\lambda E \sim \text{Exponential}(1)$.
Hint 8: If random variables $E_1, \dots, E_n$ are independent and follow the Exponential(1) distribution, then $2\sum_{i=1}^n E_i \sim \chi^2_{2n}$.

A pivot is a function of the data and the parameters of the statistical model whose distribution does not depend on the unknown parameters of the statistical model.

By *Hint 7*, $\frac{1}{\theta}X_i^3 \sim \text{Exponential}(1)$ for all i, and also these are independent random variables (because the X_i's are). Then *Hint 8* (with $n = 10$) implies that

$$\frac{2}{\theta}\sum_{i=1}^{20} X_i^3 \sim \chi^2_{20}$$

is a pivot (the distribution of the left-hand side does not depend on the unknown parameter θ, that is why it is a pivot).

(b) [5 points] Construct a confidence interval for θ of confidence level 0.9.

Using the pivot from *(a)*, we have

$$\mathbb{P}\left\{\chi^2_{20,0.05} \leq \overbrace{\frac{2}{\theta}\sum_{i=1}^{20} X_i^3}^{\text{pivot} \sim \chi^2_{20}} \leq \chi^2_{20,0.95}\right\} = 0.9.$$

which is equivalent to

$$\underbrace{\mathbb{P}\left\{\frac{2}{\chi^2_{20,0.95}}\sum_{i=1}^{20}X_i^3 \le \theta \le \frac{2}{\chi^2_{20,0.05}}\sum_{i=1}^{20}X_i^3\right\}}_{\text{CI for } \theta \text{ of level 0.9}} = 0.9.$$

(c) [5 points] It has been observed that $\overline{X^3} = 1.6$. Use *(b)* to test at significance level 0.1 whether θ deviates from 2.

We have to test $H_0\colon \theta = 2$ VS $H_1\colon \theta \neq 2$. The CI from *(b)* is of confidence level 0.9. We will use the test

$$\begin{aligned} 2 \notin \text{ the CI } &\Rightarrow \text{ reject } H_0 \\ 2 \in \text{ the CI } &\Rightarrow \text{ fail to reject } H_0, \end{aligned}$$

which has significance level 1-0.9=0.1.

We have $\sum_{i=1}^{10} X_i^3 = 10\overline{X^3} = 16$, $\chi^2_{20,0.95} = 31.41$, $\chi^2_{20,0.05} = 10.85$, and hence the confidence interval is

$$\left[\frac{2}{31.41}\cdot 16, \frac{2}{10.85}\cdot 16\right] = [1.02, 2.95].$$

Since $2 \in [1.02, 2.95]$, we fail to reject $H_0\colon \theta = 2$.

Exercise 6 [23 points (final retake); 17 points (full retake)]

Let $X_1, \ldots, X_n$ be independent random variables from the $\Gamma(3, 1/\theta)$ distribution (see Appendix 1), where $\theta > 0$ is an unknown parameter.

(a) [8 points] The maximum likelihood estimator for θ is $\widehat{\lambda}_{\text{ML}} = \overline{X}/3$. Compute the Wald confidence interval $\theta = \widehat{\lambda}_{\text{ML}} \pm \frac{1}{\sqrt{n\widehat{i_\theta}}}\Phi^{-1}(1-\alpha/2)$. What estimator for the Fisher information i_θ (see Appendix 2) do you use? What is the confidence level of this confidence interval, and is it the exact or an approximate confidence level?

First we compute the Fisher information:

$$\begin{aligned} p_\theta(x) &\overset{\text{Appendix 1}}{=} \frac{1}{\Gamma(3)\theta^3}x^2 e^{-x/\theta}, \\ \ln p_\theta(x) &= -\ln\Gamma(3) - 3\ln\theta + 2\ln x - \frac{x}{\theta}, \\ \dot{\ell}(x) &= \frac{\partial \ell}{\partial\theta}\ln p_\theta(x) = -\frac{3}{\theta} + \frac{x}{\theta^2}, \end{aligned}$$

and hence,

$$i_\theta = \text{Var}_\theta\dot{\ell}(X_1) = \text{Var}_\theta\left(-\frac{3}{\theta}+\frac{X_1}{\theta^2}\right) = \text{Var}\left(\frac{X_1}{\theta^2}\right) = \frac{1}{\theta^4}\text{Var}\underbrace{X_1}_{\sim\Gamma(3,1/\theta)} \overset{\text{Appendix 1}}{=} \frac{1}{\theta^4}\cdot\frac{3}{(1/\theta^2)} = \frac{3}{\theta^2}.$$

We use the plug-in estimator for the Fisher information

$$\widehat{i_\theta} = i_{\widehat{\theta}_{\text{ML}}} = i_{\overline{X}/3} = \frac{27}{(\overline{X})^2},$$

and get the Wald confidence interval

$$\theta = \frac{\overline{X}}{3} \pm \frac{\overline{X}}{3\sqrt{3n}}\Phi^{-1}(1-\alpha/2).$$

The confidence level of this confidence interval is approximately $1-\alpha$.

(b) [6 points (final retake only)] Does the Cramer-Rao lower bound (see Appendix 1) imply that $\overline{X}/3$ is a UMVU estimator for θ?

The estimator $\overline{X}/3$ is unbiased estimator for θ:

$$\mathbb{E}\frac{\overline{X}}{3} = \frac{1}{3}\mathbb{E}X_1 \overset{\text{Appendix 1}}{=} \frac{1}{3}\cdot\frac{3}{1/\theta} = \theta.$$

The LHS in the Cramer-Rao lower bound is

$$\mathrm{Var}\frac{\overline{X}}{3} = \frac{\mathrm{Var}X_1}{9n} \overset{\text{Appendix 1}}{=} \frac{1}{9n}\cdot\frac{3}{(1/\theta^2)} = \frac{\theta^2}{3n},$$

and the RHS in the Cramer-Rao lower bound is

$$\frac{1}{ni_\theta} \overset{(a)}{=} \frac{1}{n\cdot 3/\theta^2} = \frac{\theta^2}{3n}.$$

We have

$$\mathrm{Var}_\theta\frac{\overline{X}}{3} = \frac{1}{ni_\theta} = \frac{\theta^2}{3n},$$

i.e. the Cramer-Rao lower bound is sharp on the estimator $\overline{X}/3$, which is unbiased for θ. Hence, is does follow that $\overline{X}/3$ is a UMVU estimator for θ.

(c) [9 points] Show that $\overline{X}$ is a sufficient and complete statistic (see Appendix 2). The estimator $\frac{3n}{3n+1}\left(\frac{\overline{X}}{3}\right)^2$ is unbiased for θ^2 (do not prove the unbiasedness), is this a UMVU estimator for θ^2?

Since

$$\begin{aligned} p_\theta(\vec{x}) &= \prod_{i=1}^n p_\theta(x_i) = \prod_{i=1}^n \frac{x_i^2}{\Gamma(3)\theta^3}e^{-x_i/\theta} \\ &= \underbrace{\frac{1}{(\Gamma(3))^n\theta^{3n}}}_{c(\theta)}\cdot\underbrace{\prod_{i=1}^n x_i^2}_{h(\vec{x})}\cdot\exp\Big(\underbrace{-\frac{1}{\theta}}_{Q_1(\theta)}\cdot\underbrace{\sum_{i=1}^n X_i}_{V_1(\vec{x})}\Big), \end{aligned}$$

the joint data distributions forms a 1-dimensional exponential family and the statistic $\sum_{i=1}^n X_i$ is sufficient.

Since the set $\{Q_1(\theta)\colon \theta > 0\} = (-\infty, 0)$ does have interior points in $\mathbb{R}^1$, the statistic $\sum_{i=1}^n X_i$ is also complete.

Then the statistic $\overline{X}$ is sufficient and complete as well because $\overline{X} = \sum_{i=1}^n X_i/n$ is a 1-to-1 correspondence.

The given estimator is a function of the sufficient and complete (as shown in *(c)*) statistic $\overline{X}$ and it is an unbiased estimator for θ^2. Hence, it is UMVU.

Appendix 1: Some relevant distributions

Exponential(λ), $\lambda > 0$

The density is given by $\begin{cases} \lambda e^{-\lambda t}, & t > 0, \\ 0, & t \leq 0. \end{cases}$

The expectation is $\frac{1}{\lambda}$, the variance is $\frac{1}{\lambda^2}$.

Gamma(α,λ) distribution, $\alpha > 0$, $\lambda > 0$

The density is given by $\begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\lambda t}, & t > 0, \\ 0, & t \leq 0, \end{cases}$ where $\Gamma(\alpha)$ is the normalizing constant.

The expectation is $\frac{\alpha}{\lambda}$, the variance is $\frac{\alpha}{\lambda^2}$.

Appendix 4: Some facts from optimality theory

Fisher information Let $X_1, \ldots, X_n$ be i.i.d. with marginal p.d.f./p.m.f. $p_\theta(x)$ and $\dot{\ell}_\theta(x) := \frac{\partial}{\partial \theta} \ln p_\theta(x)$, $\ddot{\ell}_\theta(x) := \frac{\partial^2}{\partial \theta^2} \ln p_\theta(x)$. Then the Fisher information is given by

$$i_\theta := \text{Var } \dot{\ell}_\theta(X_1) = -\mathbb{E} \, \ddot{\ell}_\theta(X_1).$$

Cramer-Rao lower bound Let $X_1, \ldots, X_n$ be i.i.d. random variables from a distribution parametrized by θ. Under certain conditions, every unbiased estimator $\widehat{\theta}$ for θ satisfies

$$\text{Var } \widehat{\theta} \geq \frac{1}{n i_\theta}.$$

Exponential family of p.d.f./p.m.f.'s on $\mathbb{R}^n$:

$$p_\theta(\vec{x}) = c(\theta) h(\vec{x}) \exp\left(\sum_{j=1}^{k} Q_j(\theta) V_j(\vec{x}) \right),$$

where $\vec{x} = (x_1, \ldots, x_n)$.

Appendix 2: Table normal distribution

	0	1	2	3	4	5	6	7	8	9
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.877	0.879	0.881	0.883
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.898	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.937	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.975	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.983	0.9834	0.9838	0.9842	0.9846	0.985	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.989
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.992	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.994	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.996	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.997	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.998	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.999	0.999

Table 1: Distribution function of the standard normal distribution on the interval $[0, 4]$. The value in the table is $\Phi(x)$ for $x = a + b/100$ where a indicates the row and b indicates the column.

Appendix 5: Table t-distribution

df	0.6	0.7	0.75	0.8	0.85	0.9	0.925	0.95	0.975	0.98	0.99	0.999
9	0.26	0.54	0.7	0.88	1.1	1.38	1.57	1.83	2.26	2.4	2.82	4.3
10	0.26	0.54	0.7	0.88	1.09	1.37	1.56	1.81	2.23	2.36	2.76	4.14
11	0.26	0.54	0.7	0.88	1.09	1.36	1.55	1.8	2.2	2.33	2.72	4.02
12	0.26	0.54	0.7	0.87	1.08	1.36	1.54	1.78	2.18	2.3	2.68	3.93
13	0.26	0.54	0.69	0.87	1.08	1.35	1.53	1.77	2.16	2.28	2.65	3.85
14	0.26	0.54	0.69	0.87	1.08	1.35	1.52	1.76	2.14	2.26	2.62	3.79
15	0.26	0.54	0.69	0.87	1.07	1.34	1.52	1.75	2.13	2.25	2.6	3.73
16	0.26	0.54	0.69	0.86	1.07	1.34	1.51	1.75	2.12	2.24	2.58	3.69
17	0.26	0.53	0.69	0.86	1.07	1.33	1.51	1.74	2.11	2.22	2.57	3.65
18	0.26	0.53	0.69	0.86	1.07	1.33	1.5	1.73	2.1	2.21	2.55	3.61
19	0.26	0.53	0.69	0.86	1.07	1.33	1.5	1.73	2.09	2.2	2.54	3.58
20	0.26	0.53	0.69	0.86	1.06	1.33	1.5	1.72	2.09	2.2	2.53	3.55
21	0.26	0.53	0.69	0.86	1.06	1.32	1.49	1.72	2.08	2.19	2.52	3.53
22	0.26	0.53	0.69	0.86	1.06	1.32	1.49	1.72	2.07	2.18	2.51	3.5
23	0.26	0.53	0.69	0.86	1.06	1.32	1.49	1.71	2.07	2.18	2.5	3.48
24	0.26	0.53	0.68	0.86	1.06	1.32	1.49	1.71	2.06	2.17	2.49	3.47
25	0.26	0.53	0.68	0.86	1.06	1.32	1.49	1.71	2.06	2.17	2.49	3.45
26	0.26	0.53	0.68	0.86	1.06	1.31	1.48	1.71	2.06	2.16	2.48	3.43
27	0.26	0.53	0.68	0.86	1.06	1.31	1.48	1.7	2.05	2.16	2.47	3.42
28	0.26	0.53	0.68	0.85	1.06	1.31	1.48	1.7	2.05	2.15	2.47	3.41

Table 2: Quantiles (columns) of the t-distribution with 9 to 28 degrees of freedom (rows).

Appendix 3: Table Chi-square distribution

df	0.001	0.005	0.01	0.025	0.05	0.1	0.125	0.2	0.25	0.333	0.5
6	0.381	0.676	0.872	1.237	1.635	2.204	2.441	3.070	3.455	4.074	5.348
7	0.598	0.989	1.239	1.690	2.167	2.833	3.106	3.822	4.255	4.945	6.346
8	0.857	1.344	1.646	2.180	2.733	3.490	3.797	4.594	5.071	5.826	7.344
9	1.152	1.735	2.088	2.700	3.325	4.168	4.507	5.380	5.899	6.716	8.343
10	1.479	2.156	2.558	3.247	3.940	4.865	5.234	6.179	6.737	7.612	9.342
11	1.834	2.603	3.053	3.816	4.575	5.578	5.975	6.989	7.584	8.514	10.341
12	2.214	3.074	3.571	4.404	5.226	6.304	6.729	7.807	8.438	9.420	11.340
13	2.617	3.565	4.107	5.009	5.892	7.042	7.493	8.634	9.299	10.331	12.340
14	3.041	4.075	4.660	5.629	6.571	7.790	8.266	9.467	10.165	11.245	13.339
15	3.483	4.601	5.229	6.262	7.261	8.547	9.048	10.307	11.037	12.163	14.339
16	3.942	5.142	5.812	6.908	7.962	9.312	9.837	11.152	11.912	13.083	15.338
17	4.416	5.697	6.408	7.564	8.672	10.085	10.633	12.002	12.792	14.006	16.338
18	4.905	6.265	7.015	8.231	9.390	10.865	11.435	12.857	13.675	14.931	17.338
19	5.407	6.844	7.633	8.907	10.117	11.651	12.242	13.716	14.562	15.859	18.338
20	5.921	7.434	8.260	9.591	10.851	12.443	13.055	14.578	15.452	16.788	19.337
21	6.447	8.034	8.897	10.283	11.591	13.240	13.873	15.445	16.344	17.720	20.337
22	6.983	8.643	9.542	10.982	12.338	14.041	14.695	16.314	17.240	18.653	21.337
23	7.529	9.260	10.196	11.689	13.091	14.848	15.521	17.187	18.137	19.587	22.337
24	8.085	9.886	10.856	12.401	13.848	15.659	16.351	18.062	19.037	20.523	23.337
25	8.649	10.520	11.524	13.120	14.611	16.473	17.184	18.940	19.939	21.461	24.337

df	0.6	0.667	0.75	0.8	0.87	0.9	0.95	0.975	0.99	0.995	0.999
6	6.211	6.867	7.841	8.558	9.992	10.645	12.592	14.449	16.812	18.548	22.458
7	7.283	7.992	9.037	9.803	11.326	12.017	14.067	16.013	18.475	20.278	24.322
8	8.351	9.107	10.219	11.030	12.636	13.362	15.507	17.535	20.090	21.955	26.125
9	9.414	10.215	11.389	12.242	13.926	14.684	16.919	19.023	21.666	23.589	27.877
10	10.473	11.317	12.549	13.442	15.198	15.987	18.307	20.483	23.209	25.188	29.588
11	11.530	12.414	13.701	14.631	16.457	17.275	19.675	21.920	24.725	26.757	31.264
12	12.584	13.506	14.845	15.812	17.703	18.549	21.026	23.337	26.217	28.300	32.910
13	13.636	14.595	15.984	16.985	18.939	19.812	22.362	24.736	27.688	29.819	34.528
14	14.685	15.680	17.117	18.151	20.166	21.064	23.685	26.119	29.141	31.319	36.123
15	15.733	16.761	18.245	19.311	21.384	22.307	24.996	27.488	30.578	32.801	37.697
16	16.780	17.840	19.369	20.465	22.595	23.542	26.296	28.845	32.000	34.267	39.252
17	17.824	18.917	20.489	21.615	23.799	24.769	27.587	30.191	33.409	35.718	40.790
18	18.868	19.991	21.605	22.760	24.997	25.989	28.869	31.526	34.805	37.156	42.312
19	19.910	21.063	22.718	23.900	26.189	27.204	30.144	32.852	36.191	38.582	43.820
20	20.951	22.133	23.828	25.038	27.376	28.412	31.410	34.170	37.566	39.997	45.315
21	21.991	23.201	24.935	26.171	28.559	29.615	32.671	35.479	38.932	41.401	46.797
22	23.031	24.268	26.039	27.301	29.737	30.813	33.924	36.781	40.289	42.796	48.268
23	24.069	25.333	27.141	28.429	30.911	32.007	35.172	38.076	41.638	44.181	49.728
24	25.106	26.397	28.241	29.553	32.081	33.196	36.415	39.364	42.980	45.559	51.179
25	26.143	27.459	29.339	30.675	33.247	34.382	37.652	40.646	44.314	46.928	52.620

Table 3: Quantiles (columns) of the chi-square distribution with 6 to 25 degrees of freedom (rows).