Exercise 1 [16 points]

The manufacturer of a drug that lowers blood pressure has changed the composition of the drug with the purpose to improve its efficacy. The new and the old versions of the drug are now compared for efficacy. The old version has been prescribed to a group of 16 patients with hypertension (high blood pressure), and the new version to a group of 10 other patients with hypertension. The blood pressures $X_1, \ldots, X_{16}$ and $Y_1, \ldots, Y_{10}$ of the patients in the two groups have been measured after they have taken the respective versions of the drug. The observed measurements have the following summaries: $\overline{X} = 163$ and $\overline{Y} = 158$, $S_X = 7.8$ and $S_Y = 9.0$. Do these data give a reason to conclude that the new version of the drug is better than the old version? Carry out a suitable standard test at significance level 0.05. Report

(a) [2 points] the statistical model (all assumptions you make on the data and the parameters),

$X_1, \ldots, X_{16}$ i.i.d. $\sim N(\mu_A, \sigma^2)$, $Y_1, \ldots, Y_{10}$ i.i.d. $\sim N(\mu_B, \sigma^2)$, the two samples are independent and have the same variance σ^2.

(b) [2 points] the null and alternative hypotheses,

The new drug being better means it lowers the blood pressure more (putting this in H_1). So we want to test $H_0 : \mu_A \leq \mu_B$ VS $H_1 : \mu_A > \mu_B$.

(c) [4 points] the test statistic and its distribution when the drugs have the same efficacy,
Hint: If $X_1, \ldots, X_m \sim N(\mu_A, \sigma_A^2)$ and $Y_1, \ldots, Y_n \sim N(\mu_B, \sigma_B^2)$, all independent, then $\dfrac{(\overline{X} - \overline{Y}) - (\mu_A - \mu_B)}{\sqrt{\sigma_A^2/m + \sigma_B^2/n}} \sim N(0,1)$ and $\dfrac{(m-1)S_X^2}{\sigma_A^2} + \dfrac{(n-1)S_Y^2}{\sigma_B^2}$ has a chi-square distribution.

We apply the hint to $\mu_A = \mu_B$ and $\sigma_A^2 = \sigma_B^2 = \sigma^2$. We have

$$\underbrace{\frac{(m-1)S_X^2}{\sigma^2} + \frac{(n-1)S_Y^2}{\sigma^2}}_{\text{bottom}} \sim \chi^2_{m+n-2} \text{ and is independent from } \underbrace{\frac{\overline{X} - \overline{Y}}{\sqrt{\sigma^2/m + \sigma^2/n}}}_{\text{top}} \sim N(0,1).$$

Hence $\dfrac{\text{top}}{\sqrt{\text{bottom}/(m+n-2)}} \sim t_{m+n-2}$, where

$$\frac{\text{top}}{\sqrt{\text{bottom}/(m+n-2)}} = \frac{\overline{X} - \overline{Y}}{\sigma\sqrt{1/m + 1/n}} \Big/ \sqrt{\frac{(m-1)S_X^2 + (n-1)S_Y^2}{\sigma^2} \cdot \frac{1}{m+n-2}}$$
$$= \frac{\overline{X} - \overline{Y}}{\sqrt{1/m + 1/n}} \Big/ \sqrt{\frac{(m-1)S_X^2 + (n-1)S_Y^2}{m+n-2}}.$$

That is, the test statistic is $T = \dfrac{\overline{X} - \overline{Y}}{\sqrt{1/m + 1/n}} \Big/ \sqrt{\dfrac{(m-1)S_X^2 + (n-1)S_Y^2}{m+n-2}}$ and its distribution under $\mu_A = \mu_B$ is t_{m+n-2}.

(d) [2 points] the critical region,

Reject H_0 if $T > t_{m+n-2,1-\alpha}$, where α is the significance level.

(e) [4 points] the conclusion.

With $m = 16$, $n = 10$ and $\alpha = 0.05$,

$$T = \frac{163 - 158}{\sqrt{1/16 + 1/10}} \Big/ \sqrt{\frac{15 \cdot 7.8^2 + 9 \cdot 9^2}{24}} = 1.5 \quad \text{and} \quad t_{m+n-2,1-\alpha} = t_{24,0.95} = 1.71.$$

Since $T = 1.5 \not> t_{m+n-2} = 1.71$, we fail to reject H_0. That is, the data do not give a reason to conclude that the new version if the drug is better than the old version.

(f) [2 points] Are the data in this experiment paired or unpaired? Would you design the experiment differently in this regard and why?

The data in this experiment are unpaired. In general, paired design is preferred where objects with similar characteristic are paired. In this experiment, the patients with similar age, weight, etc. should be paired. This will reduce the variability of the data and make the difference in the drug efficacy more apparent.

Exercise 2 [15 points]

Let $X_1, \ldots, X_8$ be a sample from the normal $N(0, \sigma^2)$ distribution, and $Y_1, \ldots, Y_8$ a sample from the normal $N(0, 9\sigma^2)$ distribution, where the parameter $\sigma^2 > 0$ is unknown. The two samples are independent and the random variables within each sample are independent.

(a) [4 points] Give the definition of a pivot. Construct a non-negative pivot out of the sample $X_1, \ldots, X_8$ and σ. What is the distribution of this pivot?

A pivot is a function of the data and the parameters of the statistical model whose distribution does not depend on the unknown parameters of the statistical model.

Option 1: Since $\dfrac{X_i}{\sigma} \sim N(0,1)$ and are independent, we have

$$\underbrace{\sum_{i=1}^{8} \left(\frac{X_i}{\sigma}\right)^2}_{\text{pivot}} \sim \chi^2_8.$$

Option 2: Since $\overline{X} \sim N(0, \sigma^2/8)$, and hence $\dfrac{\overline{X}}{\sigma/\sqrt{8}} \sim N(0,1)$, we have

$$\underbrace{\left(\frac{\overline{X}}{\sigma/\sqrt{8}}\right)^2}_{\text{pivot}} \sim \chi^2_1.$$

(b) [3 points] Construct a non-negative pivot out of both samples. What is the distribution of this pivot?

Option 1: Since $\dfrac{X_i}{\sigma} \sim N(0,1)$, $\dfrac{Y_j}{3\sigma} \sim N(0,1)$, all independent, we have

$$\underbrace{\sum_{i=1}^{8} \left(\frac{X_i}{\sigma}\right)^2 + \sum_{j=1}^{8} \left(\frac{Y_j}{3\sigma}\right)^2}_{\text{pivot}} \sim \chi^2_{8+8}.$$

Option 2: Since $\overline{X} \sim N(0, \sigma^2/8)$, $\overline{Y} \sim N(0, 9\sigma^2/8)$ are independent, and hence $\dfrac{\overline{X}}{\sigma/\sqrt{8}} \sim N(0,1)$, $\dfrac{\overline{Y}}{3\sigma/\sqrt{8}} \sim N(0,1)$ are independent, we have

$$\underbrace{\left(\frac{\overline{X}}{\sigma/\sqrt{8}}\right)^2 + \left(\frac{\overline{Y}}{3\sigma/\sqrt{8}}\right)^2}_{\text{pivot}} \sim \chi^2_{1+1}.$$

(c) [3 points] Based on either of the pivots from *(a)* or *(b)* (only one of them), construct a confidence interval for σ^2 of confidence level 0.9.

Using the "Option 1" pivot from *(b)*, we have

$$\mathbb{P}\Big\{\chi^2_{16,0.05} \le \overbrace{\frac{1}{\sigma^2}\Big(\sum_{i=1}^{8} X_i^2 + \frac{1}{9}\sum_{j=1}^{8} Y_j^2\Big)}^{\text{pivot} \sim \chi^2_{16}} \le \chi^2_{16,0.95}\Big\} = 0.9.$$

which is equivalent to

$$\mathbb{P}\Big\{\underbrace{\frac{1}{\chi^2_{16,0.95}}\Big(\sum_{i=1}^{8} X_i^2 + \frac{1}{9}\sum_{j=1}^{8} Y_j^2\Big) \le \sigma^2 \le \frac{1}{\chi^2_{16,0.05}}\Big(\sum_{i=1}^{8} X_i^2 + \frac{1}{9}\sum_{j=1}^{8} Y_j^2\Big)}_{\text{confidence interval for } \sigma^2 \text{ of level } 0.9}\Big\} = 0.9.$$

(d) [2 points] Describe how you can test $H_0\colon \sigma^2 = \sigma_0^2$ VS $H_1\colon \sigma^2 \ne \sigma_0^2$ at significance level 0.1 using the confidence interval from *(c)*.

If $\sigma_0^2 \notin \Big[\frac{1}{\chi^2_{16,0.95}}\Big(\sum_{i=1}^{8} X_i^2 + \frac{1}{9}\sum_{j=1}^{8} Y_j^2\Big), \frac{1}{\chi^2_{16,0.05}}\Big(\sum_{i=1}^{8} X_i^2 + \frac{1}{9}\sum_{j=1}^{8} Y_j^2\Big)\Big]$, reject H_0.
Otherwise fail to reject H_0.

(e) [3 points] It has been observed that $\overline{X^2} = 1.3$, $\overline{Y^2} = 9.9$, $\overline{X} = 1.1$, and $\overline{Y} = 3.1$. Using *(d)*, test at significance level 0.1 whether σ^2 deviates from 1.6^2.
Hint: You may need to take into account only some of the summaries $\overline{X^2} = 1.3$, $\overline{Y^2} = 9.9$, $\overline{X} = 1.1$, and $\overline{Y} = 3.1$, not all of them.

We have to test $H_0\colon \sigma^2 = 1.6^2$ VS $H_1\colon \sigma^2 \ne 1.6^2$, so we apply the test from *(d)* to $\sigma_0^2 = 1.6^2$. We have $\sum_{i=1}^{8} X_i^2 = 8\overline{X^2}$, $\sum_{j=1}^{8} Y_j^2 = 8\overline{Y^2}$, $\chi^2_{16,0.95} = 26.3$, $\chi^2_{16,0.05} = 7.96$, and hence the confidence interval is

$$\Big[\frac{1}{26.3}(8\cdot 1.3 + 8\cdot 9.9/9), \frac{1}{7.96}(8\cdot 1.3 + 8\cdot 9.9/9)\Big] = [0.73, 2.41].$$

Since $\sigma_0^2 = 2.56 \notin [0.73, 2.41]$, we reject H_0, i.e. σ^2 does deviate from 1.6^2.

Exercise 3 [19 points]

Let $X_1, \dots, X_n$ be independent random variables from the Poisson distribution with unknown parameter $\lambda > 0$. Recall that the p.m.f. is given by $p_\lambda(x) = e^{-\lambda}\frac{\lambda^x}{x!}$, $x = 0, 1, 2, \dots$, and $\mathbb{E}X_1 = \mathrm{Var}X_1 = \lambda$.

(a) [8 points] The maximum likelihood estimator for λ is $\widehat{\lambda}_{\mathrm{ML}} = \overline{X}$, and i_λ denotes the Fisher information (see Appendix 1). What is approximately the distribution of $\sqrt{ni_\lambda}(\overline{X} - \lambda)$ for large n? Based on this approximation, construct an approximate confidence interval for λ of confidence level $1-\alpha$. What estimator for i_λ do you use?

As $n \to \infty$, $\sqrt{ni_\lambda}(\widehat{\lambda}_{\mathrm{ML}} - \lambda) = \sqrt{ni_\lambda}(\overline{X} - \lambda) \approx N(0,1)$ and also $\sqrt{n\widehat{i_\lambda}}(\overline{X} - \lambda) \approx N(0,1)$. We use the second approximate pivot (with the estimated Fisher information) to get a CI, we describe how to estimate the Fisher information below.
We have

$$\mathbb{P}\{-z_{\alpha/2} \le \sqrt{n\widehat{i_\lambda}}(\overline{X} - \lambda) \le z_{\alpha/2}\} \approx 1-\alpha,$$

which is equivalent to

$$\mathbb{P}\{\overline{X} - \frac{1}{\sqrt{n\widehat{i_\lambda}}}z_{\alpha/2} \le \lambda \le \overline{X} + \frac{1}{\sqrt{n\widehat{i_\lambda}}}z_{\alpha/2}\} \approx 1-\alpha,$$

i.e. we get the following approximate confidence interval of level $1-\alpha$ (the Wald CI),

$$\lambda = \overline{X} \pm \frac{1}{\sqrt{n\widehat{i_\lambda}}} z_{\alpha/2}.$$

Now we compute/ estimate the Fisher information:

$$\ell_\lambda(x) = \log p_\lambda(x) = -\lambda + x \ln \lambda - \ln x!, \quad \dot{\ell}(x) = \frac{\partial \ell}{\partial \lambda} \ell_\lambda(x) = -1 + \frac{x}{\lambda},$$

and hence,

$$i_\lambda = \mathbb{V}\dot{\ell}_\lambda(X_1) = \mathbb{V}\left(-1 + \frac{X_1}{\lambda}\right) = \mathbb{V}\left(\frac{X_1}{\lambda}\right) = \frac{1}{\lambda^2}\mathbb{V}X_1 = \frac{1}{\lambda}.$$

We use the plug-in estimator for the Fisher information

$$\widehat{i_\lambda} = \frac{1}{\hat{\lambda}_{\mathrm{ML}}} = \frac{1}{\overline{X}}.$$

So the Wald CI of approximate confidence level $1-\alpha$ is

$$\lambda = \overline{X} \pm \sqrt{\frac{\overline{X}}{n}} z_{\alpha/2}.$$

(b) [3 points] Does the Cramer-Rao lower bound (see Appendix 1) imply that $\overline{X}$ is a UMVU estimator for λ?

We have

$$\mathbb{V}\overline{X} = \frac{\mathbb{V}X_1}{n} = \frac{\lambda}{n}.$$

Now we compute the Cramer-Rao lower bound (CRLB) with $g(\lambda) = \lambda$ and $i_\lambda = 1/\lambda$ (from *(a)*),

$$\mathrm{CRLB} = \frac{(g'(\lambda))^2}{n i_\lambda} = \frac{1}{n/\lambda} = \frac{\lambda}{n}.$$

We have

$$\mathbb{V}\overline{X} = \mathrm{CRLB},$$

i.e. the Cramer-Rao lower bound is sharp on $\overline{X}$, and hence is does follow that $\overline{X}$ is an UMVU estimator for λ.

(c) [4 points] Show that $\overline{X}$ is a sufficient and complete statistic (see Appendix 1).

Since

$$L(\lambda) \overset{\text{i.i.d.}}{=} \prod_{i=1}^{n} p_\lambda(X_i) = e^{-\lambda n} \frac{\lambda^{\sum_{i=1}^n X_i}}{\prod_{i=1}^n X_i!} = \underbrace{e^{-\lambda n}}_{c(\lambda)} \cdot \exp\Big(\underbrace{(\log \lambda) \cdot n}_{Q_1} \cdot \underbrace{\overline{X}}_{V_1}\Big) \underbrace{\frac{1}{\prod_{i=1}^n X_i!}}_{h}.$$

the statistical model is a 1-dimensional exponential family and the statistic $\overline{X}$ is sufficient.

Since the set $\{Q_1(\lambda) \colon \lambda > 0\} = (-\infty, \infty)$ does have interior points in $\mathbb{R}^1$, the statistic $\overline{X}$ is also complete.

(d) [4 points] Find a UMVU estimator for λ^2.

We have

$$\mathbb{E}(\overline{X})^2 = \mathbb{V}\overline{X} + (\mathbb{E}\overline{X})^2 = \frac{\mathbb{V}X_1}{n} + (\mathbb{E}X_1)^2 = \frac{\lambda}{n} + \lambda^2.$$

We also have $\mathbb{E}\overline{X} = \lambda$, and hence

$$\mathbb{E}\left((\overline{X})^2 - \frac{\overline{X}}{n}\right) = \frac{\lambda}{n} + \lambda^2 - \frac{\lambda}{n} = \lambda^2.$$

That is, $(\overline{X})^2 - \dfrac{\overline{X}}{n}$ is an unbiased estimator for λ^2, and since it is a funciton of the sufficient and complete statistic $\overline{X}$, it is also UMVU.