Department of Mathematics	Partial exam I – Statistics
VU University Amsterdam	28 October 2016

The use of a (non-graphical) calculator is allowed. After correction exams can be inspected through the educational office. The exam comprises 3 exercises and 2 appendices. The rating of each exercises can be found below with the exercise. Grade = $(total + 3)/3$.

Write clearly your name, your student number and the name of your teaching assistant at the top of the page!

SUCCESS!

Exercise 1 [8 points]

Let X_1, X_2 be independent continuous random variables on $\mathbb{R}$ from the distribution F_θ with density $f_\theta(x) = \{\pi[1 + (x - \theta)^2]\}^{-1}$, where θ an unknown parameter.

a. [4 points] Determine the distribution function F_θ and the quantile function. *Hint:* Recall $\frac{d}{dy} \arctan(y) = (1 + y^2)^{-1}$ and $\lim_{y\to\pm\infty} \arctan(y) = \pm\frac{\pi}{2}$.

b. [4 points] Show that the maximum likelihood estimate of θ on the basis of the sample $X_1 = x_1, X_2 = x_2$ with $x_1 = 1$ and $x_2 = 0$ equals $\frac{1}{2}$.

Exercise 2 [8 points]

Let $X_1, \ldots, X_n$ be independent continuous random variables from the Weibull distribution (details in the appendix) with unknown scale parameter λ and known shape prameter k.

a. [1 point] Give the moment estimator of λ.

b. [4 points] Assume that the prior on λ is an inverse gamma distribution with shape parameter α and scale parameter β. Compute the corresponding Bayes estimator of λ.

c. [3 points] Assume that the posterior distribution of λ is itself an inverse gamma distribution with shape parameter $\alpha + 2$ and scale parameter $\beta + k\bar{x}$. Compute the Bayes estimator of λ^2.

Exercises 3 [11 points]

On November 8 citizens of the US go to the polls to vote for their next president (either Donald Trump or Hillary Clinton). Among VU academics only few like Trump. We suspect that VU students follow suit (*NL: het voorbeeld volgen*) and fewer than 25% of the VU students like Trump. To investigate this we ask 50 arbitrary VU-students and register 9 students liking Trump.

a. [1 point] Formulate an appropriate statistical model and suitable null (H_0) and alternative (H_a) hypothesis to investigate our suspicion.

b. [5 points] Test the null hypothesis from part a.) at (significance) level $\alpha_0 = 0.10$ (see Appendix 2). Report:

- *i)* the test statistic,
- *ii)* the distribution (by approximation) of the test statistic under the (boundary of the) null hypothesis,
- *iii)* the critical region of the test,
- *iv)* the conclusion of the test.

c. [2 points] Calculate the p-value corresponding to the test above.

d. [3 points] Assume that the critical value c_{α_0} (with level $\alpha_0 = 0.10$) of the rejection region equals $\frac{1}{4}n - \frac{1}{2} - 1.28\sqrt{n/16}$. How many VU-students should have been asked so that the probability of a type II error is smaller than $\frac{1}{5}$ at $p = \frac{1}{5}$, with p the probability of a VU-student liking Trump?

Appendix 1: Some relevant distributions

Weibull distribution

A random variable X follows a Weibull distribution with support over $\mathbb{R}_{\geq 0}$ with shape parameter $k > 0$ and scale parameter $\lambda > 0$ if it has density:

$$f_{\lambda,k}(x) = k\lambda^{-1}x^{k-1}\exp(-x^k/\lambda).$$

Its expectation and variance are $\mathbb{E}(X) = \lambda^{1/k}\Gamma[(k+1)/k]$ and $\mathbb{V}(X) = \lambda^{2/k}\{\Gamma[(k+2)/k] - \Gamma[(k+2)/k]\Gamma[(k+2)/k]\}$, respectively, where $\Gamma(\cdot)$ denoted the Gamma function. The Gamma functions satisfies $\Gamma(x+1) = x\Gamma(x)$.

Inverse-gamma distribution

A random variable X follows an inverse gamma distribution with support over $\mathbb{R}_{>0}$ with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$ if it has density:

$$f_{\alpha,\beta}(x) = \beta^{\alpha}[\Gamma(\alpha)]^{-1}x^{-\alpha-1}\exp(-\beta/x).$$

Its expectation and variance are $\mathbb{E}(X) = \frac{\beta}{\alpha-1}$ (for $\alpha > 1$) and $\mathbb{V}(X) = \frac{\beta}{(\alpha-1)^2(\alpha-2)}$ (for $\alpha > 2$), respectively.

Appendix 2: Table normal distribution

	0	1	2	3	4	5	6	7	8	9
0.0	0.5	0.504	0.508	0.512	0.516	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.591	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.648	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.67	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.695	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.719	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.758	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.791	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.834	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.877	0.879	0.881	0.883
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.898	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.937	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.975	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.983	0.9834	0.9838	0.9842	0.9846	0.985	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.989
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.992	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.994	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.996	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.997	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.998	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.999	0.999
3.1	0.999	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1	1	1	1	1	1	1	1	1	1

Table 1: Distribution function of the standard normal distribution on the interval $[0, 4]$. The value in the table is $\Phi(x)$ for $x = a + b/100$ with a the value in the first columns while b that in the first row.