Answer to question 1

Answer to question 1a

The distribution function is (using the hints provided in the exercise):

$$\begin{aligned} F_\theta(x) &= \int_{-\infty}^{x} \{\pi[1+(y-\theta)^2]\}^{-1} dy = \int_{-\infty}^{x-\theta} \{\pi[1+z^2]\}^{-1} dz \\ &= \pi^{-1}[\arctan(z)]_{-\infty}^{x-\theta} = \pi^{-1}\arctan(x-\theta) + \tfrac{1}{2}. \end{aligned}$$

This distribution function is continuous and monotone in x. Hence, the quantile function is then the inverse of $F_\theta(x)$:

$$\begin{aligned} \alpha &= F_\theta(x_\alpha) \\ \Leftrightarrow \quad \alpha &= \pi^{-1}\arctan(x_\alpha-\theta) + \tfrac{1}{2} \\ \Leftrightarrow \quad \alpha - \tfrac{1}{2} &= \pi^{-1}\arctan(x_\alpha-\theta) \\ \Leftrightarrow \quad \pi(\alpha - \tfrac{1}{2}) &= \arctan(x_\alpha-\theta) \\ \Leftrightarrow \quad \tan[\pi(\alpha - \tfrac{1}{2})] &= x_\alpha - \theta \\ \Leftrightarrow \quad \theta + \tan[\pi(\alpha - \tfrac{1}{2})] &= x_\alpha. \end{aligned}$$

Thus, the quantile function is $F^{-1}(\alpha) = \theta + \tan[\pi(\alpha - \frac{1}{2})]$.

Answer to question 1b

The likelihood is:

$$L(X_1 = 1, X_2 = 0; \theta) = \{\pi[1+(\theta-1)^2]\}^{-1}\{\pi[1+\theta^2]\}^{-1}$$

Its logarithm is the log-likelihood:

$$\mathcal{L}(X_1 = 1, X_2 = 0; \theta) \propto -\log[1+(\theta-1)^2] - \log[1+\theta^2]$$

Arrive at the likelihood equation by equating the derivate of the log-likelihood w.r.t. θ to zero:

$$0 = -\frac{2(\theta-1)}{1+(\theta-1)^2} - \frac{2\theta}{1+\theta^2}$$

This equals zero when:

$$0 = -(\theta-1)[1+\theta^2] - \theta[1+(\theta-1)^2].$$

This factorizes to:

$$0 = (2\theta-1)(\theta^2-\theta+1).$$

Hence, $\hat{\theta}_{ML} = \frac{1}{2}$ as the second factor yields imaginary roots (verify by the *abc*-formula).
It remains to verify that the ML estimate indeed maximizes the likelihood. The second order derivative of the log-likelihood with respect to θ is:

$$-\frac{2}{1+(\theta-1)^2} + \frac{4(\theta-1)^2}{[1+(\theta-1)^2]} - \frac{2}{1+\theta^2} + \frac{4\theta^2}{[1+\theta^2]^2} = \frac{-2+2(\theta-1)^2}{[1+(\theta-1)^2]} + \frac{-2+2\theta^2}{[1+\theta^2]^2}.$$

Evaluate this derivative at $\theta = \frac{1}{2}$ and note it is negative, which implies the ML estimate indeed maximizes the likelihood.

Note: To answer the exam exercise one need not find the factorization above. It suffices to verify through substitution that $\theta = \frac{1}{2}$ is a zero of the likelihood estimating equation.

Answer to Exercise 2

Answer to Exercise 2a
The first order (population) moment of the Weibull distributed X_i is $\mathbb{E}(X_i) = \lambda^{1/k}\Gamma[(k+1)/k]$. The first order sample moment is the sample mean $\bar{X} = \frac{1}{n}\sum_{i=1} X_i$. Equate the population and sample moment, solve for p and obtain: $\hat{\lambda}_{MoM} = \{\bar{X}/\Gamma[(k+1)/k]\}^k$.

Answer to Exercise 2b
Denote the inverse gamma prior on λ by π_λ. The Bayes estimator then is:

$$\mathbb{E}(p|X_1 = x_1, \ldots, X_n = x_n) \quad = \quad \int_0^\infty \lambda \, \frac{\pi(\lambda) \; P(X_1 = x, \ldots, X_n = x_n \,|\, \bar{\lambda} = \lambda)}{\int_0^\infty \pi(\lambda) \; P(X_1 = x, \ldots, X_n = x_n \,|\, \bar{\lambda} = \lambda) \; d\lambda} \, d\lambda.$$

The denominator is:

$$\begin{aligned}
&\int_0^\infty \pi(\lambda) \; P(X_1 = x, \ldots, X_n = x_n \,|\, \bar{\lambda} = \lambda) \; d\lambda \\
&\quad = \int_0^\infty \pi(\lambda) \prod_{i=1}^n P_{\lambda,k}(X_i = x_i \,|\, \bar{\lambda} = \lambda) \; d\lambda \\
&\quad = \int_0^\infty \beta^\alpha [\Gamma(\alpha)]^{-1} \lambda^{-\alpha-1} \exp(-\beta/\lambda) \prod_{i=1}^n k\lambda^{-1} x_i^{k-1} \exp(-x_i^k/\lambda) \; d\lambda \\
&\quad = \beta^\alpha [\Gamma(\alpha)]^{-1} k^n \Big(\prod_{i=1}^n x_i^{k-1} \Big) \int_0^\infty \lambda^{-\alpha-1-n} \exp\Big[-\lambda^{-1} \Big(\beta + \sum_{i=1}^n x_i^k \Big) \Big] \; d\lambda \\
&\quad = \beta^\alpha [\Gamma(\alpha)]^{-1} k^n \Big(\prod_{i=1}^n x_i^{k-1} \Big) \Big(\beta + \sum_{i=1}^n x_i^k \Big)^{-\alpha-n} \Gamma(\alpha+n) \\
&\qquad \times \int_0^\infty \Big(\beta + \sum_{i=1}^n x_i^k \Big)^{\alpha+n} [\Gamma(\alpha+n)]^{-1} \lambda^{-\alpha-1-n} \exp\Big[-\lambda^{-1} \Big(\beta + \sum_{i=1}^n x_i^k \Big) \Big] \; d\lambda \\
&\quad = \beta^\alpha [\Gamma(\alpha)]^{-1} k^n \Big(\prod_{i=1}^n x_i^{k-1} \Big) \Big(\beta + \sum_{i=1}^n x_i^k \Big)^{-\alpha-n} \Gamma(\alpha+n),
\end{aligned}$$

where in the last step the integral vanishes as the integrand is an inverse gamma density. By the same token the numerator is:

$$\begin{aligned}
&\int_0^\infty \lambda \; \pi(\lambda) \prod_{i=1}^n P_{\lambda,k}(X_i = x_i \,|\, \bar{\lambda} = \lambda) \; d\lambda \\
&\quad = \beta^\alpha [\Gamma(\alpha)]^{-1} k^n \Big(\prod_{i=1}^n x_i^{k-1} \Big) \Big(\beta + \sum_{i=1}^n x_i^k \Big)^{-\alpha-n+1} \Gamma(\alpha+n-1).
\end{aligned}$$

The Bayes estimator is then given by division of the numerator by the denominator:

$$\hat{\lambda}_B \quad = \quad \Big(\beta + \sum_{i=1}^n x_i^k \Big) \Gamma(\alpha+n-1) [\Gamma(\alpha+n)]^{-1} = \Big(\beta + \sum_{i=1}^n x_i^k \Big) (\alpha+n-1)^{-1}.$$

in which the property of the Gamma-function has been used.

Note: The argument above may be abbreviated, as the normalization constant does not effect the shape of the posterior.

Answer to Exercise 2c
In general, $\mathbb{E}(X^2) = \mathbb{V}(X) + [\mathbb{E}(X)]^2$. Furthermore, the posterior distribution of λ is with shape parameter $\alpha + 2$ and scale parameter $\beta + k\bar{x}$ (given in the exercise). The expectation and variance

of an inverse gamma distributed random variable are provided in the appendix. Hence, $\mathbb{E}(X) = \frac{\beta+k\bar{x}}{\alpha+1}$ and $\mathbb{V}(X) = \frac{\beta+k\bar{x}}{\alpha(\alpha+1)^2}$. Then:

$$\begin{aligned}\mathbb{E}(\lambda^2|X_1 = x_1, \ldots, X_n = x_n) &= \mathbb{V}(\lambda|X_1 = x_1, \ldots, X_n = x_n) + [\mathbb{E}(\lambda|X_1 = x_1, \ldots, X_n = x_n)]^2 \\ &= \tfrac{\beta+k\bar{x}}{\alpha(\alpha+1)^2} + \tfrac{(\beta+k\bar{x})^2}{(\alpha+1)^2} = \tfrac{\beta+k\bar{x}+\alpha(\beta+k\bar{x})^2}{\alpha(\alpha+1)^2}.\end{aligned}$$

Answer to Exercise 3

Answer to Exercise 3a

The statistical model is the Binomial distribution $Bin(n, p)$ with $n = 50$ and $p \in [0, 1]$, the probability of liking Trump. The hypothesis $H_0 : p \geq \frac{1}{4}$ vs. $H_a : p < \frac{1}{4}$.

Answer to Exercise 3b

The test statistic $T = X$ the number of Trump dislikes. Then, $T \sim_{H_0} Bin(n, \frac{1}{4})$. The Binomial distribution may be approximated by the normal: $T \approx \mathcal{N}(\frac{1}{4}n, \frac{3n}{16})$ for $np(1-p) > 5$ (which is satisfied here). The critical region at level $\alpha_0 = 0.10$ is $K = \{0, \ldots, c_{\alpha_0}\}$. Now derive the critical value c_{α_0} from the 2^{nd} convention (i.e. choose a test of level α_0):

$$\begin{aligned}P_{H_0}(T \in K) &= P_{p=\frac{1}{4}}(T \leq c_{\alpha_0} + \tfrac{1}{2}) = P_{p=\frac{1}{4}}\left(\frac{T-\frac{1}{4}n}{\sqrt{\frac{3n}{16}}} \leq \frac{c_{\alpha_0}+\frac{1}{2}-\frac{1}{4}n}{\sqrt{\frac{3n}{16}}}\right) \\ &\approx \Phi_{0,1}\left(\frac{c_{\alpha_0}+\frac{1}{2}-\frac{1}{4}n}{\sqrt{\frac{3n}{16}}}\right) = 0.10 = \Phi_{0,1}(-1.28),\end{aligned}$$

in which the continuity correction has been applied. Apply $\Phi_{0,1}^{-1}$, obtain $c_{\alpha_0} + \frac{1}{2} - \frac{1}{4}n = -1.28\sqrt{\frac{3n}{16}}$, and solve for c_{α_0}: $c_{\alpha_0} = \frac{1}{4}n - \frac{1}{2} - 1.28\sqrt{\frac{3n}{16}}$. Substitute n to arrive at $c_{\alpha_0} = 8.08$. Hence, $K = \{0, \ldots, 8\}$. As we have observed 9 Trump likes, the null hypothesis is not rejected.

Answer to Exercise 3c

Evaluate the p-value:

$$p = P_{p=\frac{1}{4}}(T \leq t) = P_{p=\frac{1}{4}}(T \leq 9 + \tfrac{1}{2}) \approx \Phi_{0,1}\left(\frac{9\frac{1}{2}-12\frac{1}{2}}{3.0612}\right) = \Phi(-0.9798) = 0.1636.$$

Answer to Exercise 3d

Subject to level $\alpha_0 = 0.10$, choose n such that $\pi(0.2; K) \geq 0.8$. Evaluate the power at $p = \frac{1}{5}$:

$$\begin{aligned}\pi(0.2; K) = P_{p=\frac{1}{5}}(T \in K) &= P_{p=\frac{1}{5}}(T \leq c_{\alpha_0}) = P_{p=\frac{1}{5}}(T \leq c_{\alpha_0} + \tfrac{1}{2}) \\ &= P_{p=0.2}\left(\frac{T-\frac{1}{5}n}{\sqrt{\frac{4n}{25}}} \leq \frac{c_{\alpha_0}+\frac{1}{2}-\frac{1}{5}n}{\sqrt{\frac{4n}{25}}}\right) = \Phi_{0,1}\left(\frac{c_{\alpha_0}+\frac{1}{2}-\frac{1}{5}n}{\sqrt{\frac{4n}{25}}}\right) \\ &= 0.80 = \Phi_{0,1}(0.8416).\end{aligned}$$

Apply $\Phi_{0,1}^{-1}$ and substitute $c_{\alpha_0} = \frac{1}{4}n - \frac{1}{2} - 1.28\frac{1}{4}\sqrt{n}$ to obtain

$$\tfrac{1}{4}n - \tfrac{1}{2} - 1.28\tfrac{1}{4}\sqrt{n} + \tfrac{1}{2} - \tfrac{1}{5}n = \tfrac{2}{5}\sqrt{n} \times 0.8416.$$

Simplified:

$$0 = \tfrac{1}{20}n - 0.65664\sqrt{n} \iff 20\ \sqrt{n}(\sqrt{n} - 13.1328) = 0.$$

Hence, $n = 172.47$.