Final Exam Statistical Methods

Vrije Universiteit Amsterdam, Faculty of Exact Science

8:30 – 11:15h, December 20, 2022

- In general: **always motivate your answers.**
- Write your answers in English.
- Only the use of a simple, non-graphical, non-programmable calculator is allowed.
- It is allowed to use scrap paper handed out by the invigilators; ask for more if needed.
- On the last three pages of this exam, some formulas and tables that you may want to use can be found.
- The total number of points you can receive is 60; Grade $= 1 + \frac{\text{points}}{6\frac{2}{3}}$.
- The division of points per question and subparts is as follows:

Question	1	2	3	4	5
Part a)	2	2	2	3	1
Part b)	2	3	8	3	3
Part c)	3	7	2	7	4
Part d)		2			6
Total	7	14	12	13	14

- Always write down the formula that you are using *before* you insert the numbers.
- If you are asked to perform a test, do not only give the conclusion of your test, but report:
  1. the hypotheses in terms of the population parameter of interest;
  2. the significance level;
  3. the test statistic and its distribution under the null hypothesis;
     (for this, also check the conditions for the test to be valid!)
  4. the observed value of the test statistic;
  5. the P-value or the critical value(s)
     (unless otherwise asked of you, you are free to choose);
  6. whether or not the null hypothesis is rejected and why;
  7. finally, phrase your conclusion in terms of the context of the problem.

Good luck!

1. In The Netherlands, there have been many discussions about how to celebrate Sinterklaas in the last couple of years. Still, many people think the Sinterklaasfeest is an important part of Dutch tradition: among 1600 randomly surveyed Dutch citizens, 1090 replied that they feel that way.

   a) Describe and justify the sampling distribution of the random variable $\hat{P}_{1600}$ which models the sample proportion of Dutch citizens who feel that Sinterklaas is an important part of Dutch tradition.

   b) Based on the sample, derive the 95% confidence interval for the population proportion of Dutch citizens who feel that Sinterklaas is an important part of Dutch tradition.

   c) In order to prepare another study about the popularity of Sinterklaas in 2023, we would like to ensure that the 95% confidence interval has a width of at most 0.04. Find the smallest sample size for which this is guaranteed.

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2. Obesity is an increasing social and medical challenge. Medical doctor L. de Heide does research into appetite-reducing medication in the northern region of Friesland. The weight of adult males who are admitted to the hospital where he works is approximately normally distributed with mean 112 kilo's and standard deviation 10 kilo's.

   a) A group of three adult males who are admitted to dr. De Heide's hospital is randomly selected and their weight is measured. Compute the probability that the mean weight of this group is higher than 130 kilo's.

   A diabetes drug that decreases the appetite - metformin - has been prescribed to a group of 35 patients who are severly overweight. The weight of these 35 patients was measured after 12 months use of metformin, and stored in the dataset X_1. The patients had a mean weight of $\bar{x}_1 = 158$ kilo's with a standard deviation of $s_1 = 7.4$ kilo's. Dr. De Heide compared this to another group of 35 patients who are severely overweight but who had not used metformin, of which the weight was measured and stored in dataset X_2. These patients had a mean weight of $\bar{x}_2 = 163$ kilo's with a standard deviation of $s_2 = 6.2$ kilo's. The two treatment groups were obtained by randomization.

   b) What hypothesis test should be used to investigate whether metformine is effective for weight loss? State the requirements of this test. Are the requirements met?

   c) Perform the hypothesis test from part b). Take significance level $\alpha = 0.01$. (See the first page of this exam for detailed instructions about testing.)

   d) Are the data in this experiment paired or unpaired? Would you design the experiment differently in this regard and why (not)?

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3. We wish to analyze differences between Computer Science (CS) and Artificial Intelligence (AI) students with regard to their fondness for programming. Suppose for simplicity that each student either likes or dislikes writing computer programs. Let p_1 be the proportion of CS students that like to program, and let p_2 be the proportion of AI students that like to program. Suppose we randomly asked 40 CS students and 36 AI students. Thereof, 30 CS students and 24 AI students replied that they like programming.

   a) Find a 90% confidence interval for the difference $p_1 - p_2$ in the population proportions of students who like programming.

   b) The teacher of the CS course on Statistical Methods claims that CS students are more fond of programming than AI students. Investigate this claim by means of an hypothesis test; use the significance level $\alpha = 5\%$. (See the first page of this exam for detailed instructions about testing.)

   c) Argue whether the critical value method and the p-value method would result in the same test outcome in b).

4. Sinc Elon Musk took over, many people left twitter and opened accounts on other platforms such as Mastodon. Many of them however did not delete their twitter account. Some of them just continued posting tweets (we call them *active users*), others stopped tweeting but kept reading other people's tweets (we call them *passive users*). By means of a suitable chi-squared test, we want to investigate whether the proportions of active vs. passive users changed after Elon Musk's take-over. The contingency table below shows the number of twitter users surveyed before and after the take-over and categorized into active and passive users.

	active users	passive users	
before take-over	526	396	922
after take-over	537	504	1041
	1063	900	1963

a) Argue whether this concerns a chi-squared test for homogeneity or a test for independence of variables? Formulate the null and alternative hypothesis.

b) Compute the expected frequencies table under the assumption that the null hypothesis is true, and show that the requirements of the test are met.

c) Perform the test from part a). Take significance level $\alpha = 0.01$. (See the first page of the exam for detailed instructions about testing.)

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5. A docent at the VU likes the use of online polls during his lectures to organize short quizzes about the lecture content. Even though neither all students physically attend the lectures nor do all attending students take part in the quizzes, the teacher is still interested in the relationship between the total quiz score of the students and their overall course grades. So his sample consists of a random selection of 38 students who attended the lectures and took part in all Mentimeter quizzes and in the exams. The following statistics summarize the sample numerically:

average quiz score $\overline{x}_{38} = 3991.751$, average overall grade $\overline{y}_{38} = 7.673$,
sample standard deviations of quiz scores, $s_x = 196.544$, and of overall grades, $s_y = 1.011$,
sample linear correlation coefficient between scores and overall grades, $r = 0.393$,
estimated standard deviation of the intercept estimator b_0 in a linear regression model, $s_{b_0} = 3.148$.
estimated standard deviation of the slope estimator b_1 in a linear regression model, $s_{b_1} = 0.00079$.

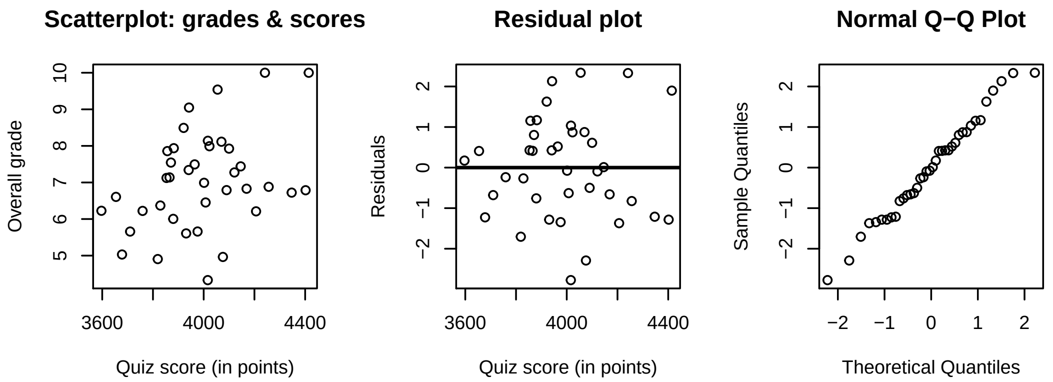


Figure 1: Scatterplot, residual plot, and normal QQ-plot for the Grades and Scores data.

a) How much of the variation (in %) among the overall grades can be explained by the quiz scores?

b) Calculate estimates of the coefficients in a linear regression model for explaining the overall grades by means of the quiz scores.
Also present the resulting linear regression equation, including the error term and its description.

c) In order to test within a linear regression model whether the quiz scores have an influence on the overall grades, several assumptions need to be checked. State which assumptions in general can be checked with the help of Residual and QQ-plots and use the middle and right plot of Figure 1 to judge whether these assumptions are satisfied for the given dataset.

d) Use a linear models-based hypothesis test to check whether quiz scores are related to overall grades. Use $\alpha = 5\%$. (See the first page of the exam for detailed instructions about testing.)

Formulas and Tables for Exam Statistical Methods

One sample testing

(i) If σ is known and certain requirements are met, the test statistic

$$Z = \frac{\overline{X}_n - \mu_0}{\sigma/\sqrt{n}}$$

has a standard normal distribution under the null hypothesis.
The margin of error for the $(1-\alpha)$ % confidence interval for the true population mean μ is $E = z_{\alpha/2}\sigma/\sqrt{n}$.
(ii) If σ is unknown and certain requirements are met, the test statistic

$$T = \frac{\overline{X}_n - \mu_0}{S_n/\sqrt{n}}$$

has a t-distribution with $n-1$ degrees of freedom under the null hypothesis.
The margin of error for the $(1-\alpha)$ % confidence interval for the true population mean μ is $E = t_{n-1,\alpha/2}s_n/\sqrt{n}$.
(iii) If certain requirements are met, the test statistic

$$Z = \frac{\hat{P}_n - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

has a standard normal distribution under the null hypothesis.
The margin of error for the $(1-\alpha)$ % confidence interval for the true population proportion p is $E = z_{\alpha/2}\sqrt{\hat{p}_n(1-\hat{p}_n)}/\sqrt{n}$.

Two independent samples

(i) If σ_1 and σ_2 are unknown, $\sigma_1 \neq \sigma_2$, and certain requirements are met, the test statistic

$$T_2 = \frac{(\overline{X}_1 - \overline{X}_2) - d_0}{\sqrt{S_1^2/n_1 + S_2^2/n_2}},$$

where d_0 is the hypothetical difference of μ_1 and μ_2 under H_0, has a t-distribution with approximately $\tilde{n}$ degrees of freedom under the null hypothesis.
We use the conservative estimate $\tilde{n} = \min\{n_1 - 1, n_2 - 1\}$.
The margin of error for the $(1-\alpha)$ % confidence interval for the true difference in population means, $\mu_1 - \mu_2$, is $E = t_{\tilde{n},\alpha/2}\sqrt{s_1^2/n_1 + s_2^2/n_2}$.
(ii) If σ_1 and σ_2 are unknown, $\sigma_1 = \sigma_2$, and certain requirements are met, the test statistic

$$T_2^{\text{eq}} = \frac{(\overline{X}_1 - \overline{X}_2) - d_0}{\sqrt{S_p^2/n_1 + S_p^2/n_2}}$$

where d_0 is the hypothetical difference of μ_1 and μ_2 under H_0, has a t-distribution with $n_1 + n_2 - 2$ degrees of freedom under the null hypothesis.
Here S_p^2 is the pooled sample variance given by

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}.$$

The margin of error for the $(1-\alpha)$ % confidence interval for the true difference in population means,

$\mu_1 - \mu_2$, is $E = t_{n_1+n_2-2,\alpha/2}\sqrt{s_p^2/n_1 + s_p^2/n_2}$.
(iii) If certain requirements are met, the test statistic

$$Z_p = \frac{(\hat{P}_1 - \hat{P}_2) - d_0}{\sqrt{\overline{P}(1-\overline{P})/n_1 + \overline{P}(1-\overline{P})/n_2}},$$

where d_0 is the hypothetical difference of p_1 and p_2 under H_0, approximately has a standard normal distribution under the null hypothesis.
Here $\overline{P} = (X_1 + X_2)/(n_1 + n_2)$ is the pooled sample proportion.
The margin of error for the $(1-\alpha)$ % confidence interval for the true difference in population proportions, $p_1 - p_2$, is $E = z_{\alpha/2}\sqrt{\hat{p}_1(1-\hat{p}_1)/n_1 + \hat{p}_2(1-\hat{p}_2)/n_2}$.

Two dependent samples

If certain requirements are met, the test statistic

$$T_d = \frac{\overline{D} - d_0}{S_d/\sqrt{n}},$$

where d_0 is the hypothetical difference of μ_1 and μ_2 under H_0, has a t-distribution with $n-1$ degrees of freedom under the null hypothesis.
The margin of error for the $(1-\alpha)$ % confidence interval for the true difference in population means, $\mu_1 - \mu_2$, is $E = t_{n-1,\alpha/2}\sqrt{s_d^2/n}$.

Correlation

If certain requirements are me, the test statistic

$$T_\rho = \frac{R}{\sqrt{(1-R^2)/(n-2)}}$$

has a t-distribution with $n-2$ degrees of freedom under the null hypothesis that the population correlation is equal to zero.

Linear regression

Let b_0 and b_1 respectively be the corresponding estimators of the unknown intercept and slope of a linear regression model with one explanatory variable. Then b_0 and b_1 are given by

$$b_1 = r\frac{s_y}{s_x} \quad \text{and} \quad b_0 = \bar{y} - b_1\bar{x}.$$

If certain requirements are met, then the test statistic

$$T_\beta = \frac{b_1}{s_{b_1}}$$

has a t-distribution with $n-2$ degrees of freedom under the null hypothesis that the population slope is equal to zero.

Goodness-of-Fit or Test of Independence/Homogeneity

If there are k different categories, or the contingency table has r rows and c columns, and certain requirements are met, then the test statistic

$$X^2 = \sum_i \frac{(O_i - E_i)^2}{E_i}$$

approximately has a chi-square distribution with $k-1$ or $(r-1)(c-1)$ degrees of freedom, respectively, under the null hypothesis.

NEGATIVE *z* Scores

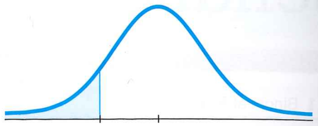


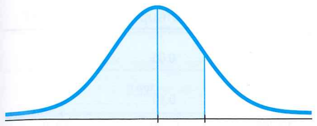
Table 2 Standard Normal (*z*) Distribution: Cumulative Area from the LEFT

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.50 and lower	.0001									
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051 *	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505 *	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
−0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
−0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

NOTE: For values of *z* below 3.49, use 0.0001 for the area.
*Use these common values that result from interpolation:

z Score	Area
−1.645	0.0500
−2.575	0.0050

(continued)



POSITIVE *z* Scores

Table 2 *(continued)* Cumulative Area from the LEFT

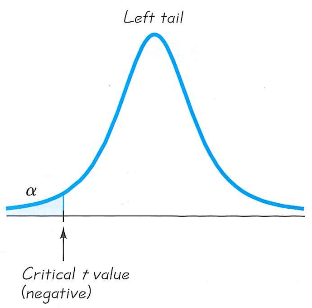
z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495 *	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949 *	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998
3.50 and up	.9999									

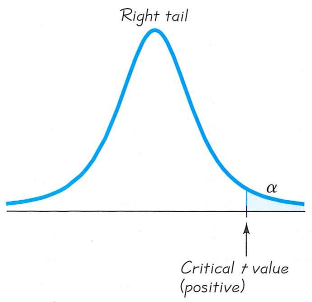
NOTE: For values of *z* above 3.49, use 0.9999 for the area.
*Use these common values that result from interpolation:

z Score	Area
1.645	0.9500
2.575	0.9950

Common Critical Values

Confidence Level	Critical Value
0.90	1.645
0.95	1.96
0.99	2.575





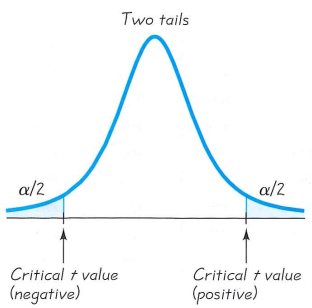


Table 3 *t* Distribution: Critical *t* Values

	Area in One Tail				
	0.005	0.01	0.025	0.05	0.10
	Area in Two Tails				
Degrees of Freedom	0.01	0.02	0.05	0.10	0.20
1	**63.657**	31.821	**12.706**	6.314	3.078
2	**9.925**	6.965	**4.303**	2.920	1.886
3	**5.841**	4.541	**3.182**	2.353	1.638
4	**4.604**	3.747	**2.776**	2.132	1.533
5	**4.032**	3.365	**2.571**	2.015	1.476
6	**3.707**	3.143	**2.447**	1.943	1.440
7	**3.499**	2.998	**2.365**	1.895	1.415
8	**3.355**	2.896	**2.306**	1.860	1.397
9	**3.250**	2.821	**2.262**	1.833	1.383
10	**3.169**	2.764	**2.228**	1.812	1.372
11	**3.106**	2.718	**2.201**	1.796	1.363
12	**3.055**	2.681	**2.179**	1.782	1.356
13	**3.012**	2.650	**2.160**	1.771	1.350
14	**2.977**	2.624	**2.145**	1.761	1.345
15	**2.947**	2.602	**2.131**	1.753	1.341
16	**2.921**	2.583	**2.120**	1.746	1.337
17	**2.898**	2.567	**2.110**	1.740	1.333
18	**2.878**	2.552	**2.101**	1.734	1.330
19	**2.861**	2.539	**2.093**	1.729	1.328
20	**2.845**	2.528	**2.086**	1.725	1.325
21	**2.831**	2.518	**2.080**	1.721	1.323
22	**2.819**	2.508	**2.074**	1.717	1.321
23	**2.807**	2.500	**2.069**	1.714	1.319
24	**2.797**	2.492	**2.064**	1.711	1.318
25	**2.787**	2.485	**2.060**	1.708	1.316
26	**2.779**	2.479	**2.056**	1.706	1.315
27	**2.771**	2.473	**2.052**	1.703	1.314
28	**2.763**	2.467	**2.048**	1.701	1.313
29	**2.756**	2.462	**2.045**	1.699	1.311
30	**2.750**	2.457	**2.042**	1.697	1.310
31	**2.744**	2.453	**2.040**	1.696	1.309
32	**2.738**	2.449	**2.037**	1.694	1.309
33	**2.733**	2.445	**2.035**	1.692	1.308
34	**2.728**	2.441	**2.032**	1.691	1.307
35	**2.724**	2.438	**2.030**	1.690	1.306
36	**2.719**	2.434	**2.028**	1.688	1.306
37	**2.715**	2.431	**2.026**	1.687	1.305
38	**2.712**	2.429	**2.024**	1.686	1.304
39	**2.708**	2.426	**2.023**	1.685	1.304
40	**2.704**	2.423	**2.021**	1.684	1.303
45	**2.690**	2.412	**2.014**	1.679	1.301
50	**2.678**	2.403	**2.009**	1.676	1.299
60	**2.660**	2.390	**2.000**	1.671	1.296
70	**2.648**	2.381	**1.994**	1.667	1.294
80	**2.639**	2.374	**1.990**	1.664	1.292
90	**2.632**	2.368	**1.987**	1.662	1.291
100	**2.626**	2.364	**1.984**	1.660	1.290
200	**2.601**	2.345	**1.972**	1.653	1.286
300	**2.592**	2.339	**1.968**	1.650	1.284
400	**2.588**	2.336	**1.966**	1.649	1.284
500	**2.586**	2.334	**1.965**	1.648	1.283
1000	**2.581**	2.330	**1.962**	1.646	1.282
2000	**2.578**	2.328	**1.961**	1.646	1.282
Large	**2.576**	2.326	**1.960**	1.645	1.282

Table 4 Chi-Square (χ^2) Distribution

	Area to the Right of the Critical Value									
Degrees of Freedom	0.995	0.99	0.975	0.95	0.90	0.10	0.05	0.025	0.01	0.005
1	—	—	0.001	0.004	0.016	2.706	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	1.610	9.236	11.071	12.833	15.086	16.750
6	0.676	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	6.304	18.549	21.026	23.337	26.217	28.299
13	3.565	4.107	5.009	5.892	7.042	19.812	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	9.312	23.542	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	10.085	24.769	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	10.865	25.989	28.869	31.526	34.805	37.156
19	6.844	7.633	8.907	10.117	11.651	27.204	30.144	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	12.443	28.412	31.410	34.170	37.566	39.997
21	8.034	8.897	10.283	11.591	13.240	29.615	32.671	35.479	38.932	41.401
22	8.643	9.542	10.982	12.338	14.042	30.813	33.924	36.781	40.289	42.796
23	9.260	10.196	11.689	13.091	14.848	32.007	35.172	38.076	41.638	44.181
24	9.886	10.856	12.401	13.848	15.659	33.196	36.415	39.364	42.980	45.559
25	10.520	11.524	13.120	14.611	16.473	34.382	37.652	40.646	44.314	46.928
26	11.160	12.198	13.844	15.379	17.292	35.563	38.885	41.923	45.642	48.290
27	11.808	12.879	14.573	16.151	18.114	36.741	40.113	43.194	46.963	49.645
28	12.461	13.565	15.308	16.928	18.939	37.916	41.337	44.461	48.278	50.993
29	13.121	14.257	16.047	17.708	19.768	39.087	42.557	45.722	49.588	52.336
30	13.787	14.954	16.791	18.493	20.599	40.256	43.773	46.979	50.892	53.672
40	20.707	22.164	24.433	26.509	29.051	51.805	55.758	59.342	63.691	66.766
50	27.991	29.707	32.357	34.764	37.689	63.167	67.505	71.420	76.154	79.490
60	35.534	37.485	40.482	43.188	46.459	74.397	79.082	83.298	88.379	91.952
70	43.275	45.442	48.758	51.739	55.329	85.527	90.531	95.023	100.425	104.215
80	51.172	53.540	57.153	60.391	64.278	96.578	101.879	106.629	112.329	116.321
90	59.196	61.754	65.647	69.126	73.291	107.565	113.145	118.136	124.116	128.299
100	67.328	70.065	74.222	77.929	82.358	118.498	124.342	129.561	135.807	140.169

Source: Donald B. Owen, *Handbook of Statistical Tables.*

Degrees of Freedom

$n - 1$	**Confidence interval or hypothesis test** for a standard deviation σ or variance σ^2
$k - 1$	**Goodness-of-fit** with *k* categories
$(r - 1)(c - 1)$	**Contingency table** with *r* rows and *c* columns
$k - 1$	**Kruskal-Wallis test** with *k* samples