Faculty of Sciences	X_400621: Sets and Combinatorics Retake Exam
Vrije Universiteit Amsterdam	Tuesday, April 4, 2023 (18:45-21:00)

This exam consists of 5 questions and a total of 90 points can be obtained. The grade is calculated as (number of points + 10)/10. Use of a calculator is not allowed.

Please provide a valid argument for every question!

1. In this exercise we consider the set-theoretic equality

$$(B\backslash C) \cup (B \cap A) = B\backslash(C\backslash A).$$

(a) [7 points] Use Venn-diagrams to show that this equality holds for arbitrary sets $A, B, C \subset U$.

(b) [9 points] Now prove the above equality, either by using the algebra of sets, or by formal reasoning (your choice).

`Note: For a proof given by the algebra of sets, indicate the laws used at each step.`

2. In the universe $U := [0, \infty)$ (the non-negative real numbers) consider the sets A_k, for $k \in \mathbb{N}$, given by

$$A_k := \left(3 - \frac{2}{k}, 4 + \frac{5}{6^k}\right].$$

(a) [5 points] Write A_k^c as a union of two intervals.

(b) [5 points] Is the sequence of sets $A_1, A_2, A_3, \ldots$: an increasing sequence? a decreasing sequence?

Note: A sequence of sets $A_1, A_2, \ldots$ is increasing if $A_1 \subset A_2 \subset A_3 \subset \ldots$ and decreasing if $A_1 \supset A_2 \supset A_3 \supset \ldots$.

(c) [8 points] Determine a simple explicit expression for the set $\bigcap_{k=1}^{\infty} A_k$.

3. In this exercise we consider license plates consisting of four successive letters (each being a capital letter from the English alphabet, so one of A, B, C, $\cdots$, Z), followed by 6 successive digits (each being one of $0, 1, 2, \cdots, 9$); e.g. SETS 002023.

(a) [7 point] How many different license plates are possible that do not start with the letter A?

(b) [9 points] How many different license plates are possible where all the letters are the same and exactly two different digits occur, with the smaller digit occurring exactly four times (so the larger digit occurs exactly two times, e.g. AAAA 744474)?

(c) [11 points] How many different license plates are possible where no letter occurs more than once and exactly three different digits occur, with these digits occurring 1 time, 2 times, and 3 times respectively (e.g. COMB 338328)? Do *not* leave any (binomial/)multinomial coefficient in the final answer.

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4. [15 points] Using mathematical induction, prove that for all $n \in \mathbb{N}$:

$$2^{2n+1} - 5 \cdot 10^n \text{ is divisible by } 6.$$

5. Let $f : [-\pi, \pi]^2 \to \mathbb{R}^2$ be the function given by

$$f(x, y) := (\cos(2x), 3\sin(y))$$

and consider the sets $A := \left(0, \frac{\pi}{2}\right) \times \left(-\frac{\pi}{2}, \pi\right)$ and $B := [-1, 0) \times (0, 3]$.

(a) [7 points] Determine the image of A under f.

(b) [7 points] Determine the inverse image of B under f.