Faculty of Sciences	X_400621: Sets and Combinatorics Retake Exam
Vrije Universiteit Amsterdam	Tuesday, April 4, 2023 (18:45-21:00)

This exam consists of 5 questions and a total of 90 points can be obtained. The grade is calculated as (number of points + 10)/10. Use of a calculator is not allowed.

Please provide a valid argument for every question!

1. In this exercise we consider the set-theoretic equality

$$(B\backslash C) \cup (B \cap A) = B\backslash(C\backslash A).$$

(a) [7 points] Use Venn-diagrams to show that this equality holds for arbitrary sets $A, B, C \subset U$.

Solution: Draw them...
3 point for the left hand side (1 point for each of these: $(B\backslash C)$, $(B \cap A)$, $(B\backslash C) \cup (B \cap A)$) and 3 points for the right hand side (1 points for $C\backslash A$, 2 more points for $B\backslash(C\backslash A)$). 1 point for the (possibly implicit) conclusion that the sides agree.

(b) [9 points] Now prove the above equality, either by using the algebra of sets, or by formal reasoning (your choice).
`Note: For a proof given by the algebra of sets, indicate the laws used at each step.`
Solution:

- Using algebra of sets:

$$\begin{aligned}(B\backslash C) \cup (B \cap A) &\overset{def.\ of\ \backslash}{=} (B \cap C^c) \cup (B \cap A) \text{ [1 point]}\\ &\overset{distributivity}{=} B \cap (C^c \cup A) \text{ [2 points]}\\ &\overset{involution}{=} B \cap (C^c \cup (A^c)^c) \text{ [2 points]}\\ &\overset{De\ Morgan}{=} B \cap (C \cap A^c)^c \text{ [2 points]}\\ &\overset{def.\ of\ \backslash}{=} B \cap (C\backslash A)^c \text{ [1 point]}\\ &\overset{def.\ of\ \backslash}{=} B\backslash(C\backslash A) \text{ [1 point]}\end{aligned}$$

  Minus points if names are missing/incorrect: -0.5 points for each missing name, where missing/incorrect 'def. of \' can only result in -0.5 points once (so never more than 2 points subtracted in total for missing/incorrect names).

- By formal reasoning: gives 6 analogous steps, with analogous points (disregarding names).

2. In the universe $U := [0, \infty)$ (the non-negative real numbers) consider the sets A_k, for $k \in \mathbb{N}$, given by

$$A_k := \left(3 - \frac{2}{k}, 4 + \frac{5}{6^k}\right].$$

(a) [5 points] Write A_k^c as a union of two intervals.

Solution:
$$A_k^c = \left[0, 3 - \frac{2}{k}\right] \cup \left(4 + \frac{5}{6^k}, \infty\right)$$
0.5 points for each boundary point (4 times) and 0.5 points for each bracket (4 times). 1 point for taking the union of them and writing down the whole expression.

(b) [5 points] Is the sequence of sets $A_1, A_2, A_3, \ldots$: an increasing sequence? a decreasing sequence?

`Note: A sequence of sets` $A_1, A_2, \ldots$ `is increasing if` $A_1 \subset A_2 \subset A_3 \subset \ldots$ `and decreasing if` $A_1 \supset A_2 \supset A_3 \supset \ldots$.

Solution: If k increases, then the left boundary point of A_k (i.e. $3 - \frac{2}{k}$) (strictly) increases and the right boundary point of A_k (i.e. $4 + \frac{5}{6^k}$) (strictly) decreases [3 points]. Therefore, the intervals A_k are getting (strictly) smaller as k increases, so the sequence of sets $A_1, A_2, \ldots$ is decreasing, and not increasing [2 points].

(c) [8 points] Determine a simple explicit expression for the set $\bigcap_{k=1}^{\infty} A_k$.

Solution:
$$\bigcap_{k=1}^{\infty} A_k = [3, 4]$$
Since $A_k \supset [3, 4]$ for all $k \in \mathbb{N}$, we have $\bigcap_{k=1}^{\infty} A_k \supset [3, 4]$ [2 points]. Further, any $x > 4$ cannot be contained in $\bigcap_{k=1}^{\infty} A_k$, as $x > 4 + 5/(6^k)$ for large enough k; similarly, any $x < 3$ cannot be contained in $\bigcap_{k=1}^{\infty} A_k$, as $x \leq 3 - 2/k$ for large enough k (this all means that $\bigcap_{k=1}^{\infty} A_k \subset [3, 4]$) [3 points]. We conclude that $\bigcap_{k=1}^{\infty} A_k = [3, 4]$ [3 points, -1 point per wrong bracket].

3. In this exercise we consider license plates consisting of four successive letters (each being a capital letter from the English alphabet, so one of A, B, C, $\cdots$, Z), followed by 6 successive digits (each being one of $0, 1, 2, \cdots, 9$); e.g. SETS 002023.

(a) [7 point] How many different license plates are possible that do not start with the letter A?

Solution: As there are (independently) 25 different possibilities for the first letter, 26 possibilities for each of the next 3 letters, and 10 possibilities for each of the 6 digits [3 points], there are (by the rule of the product) $25 \cdot 26^3 \cdot 10^6$ different license plates possible [4 points].

(b) [9 points] How many different license plates are possible where all the letters are the same and exactly two different digits occur, with the smaller digit occurring exactly four times (so the larger digit occurs exactly two times, e.g. AAAA 744474)?

Solution: There are 26 possible ways to choose the 4 (all equal) letters [1 point]. There are $\binom{6}{4} (= \binom{6}{2})$ possible ways to choose the 4 places (without order) for the digit occurring 4 times, fixing 2 (unordered) places for the other digit [2 points]. There are $\binom{10}{2} (= 10 \cdot 9/2)$ possible ways to choose a smallest and largest digit [2 points, only one point if the factor 2 is missing, i.e. for the answer $10 \cdot 9$]. So a total combination of $26 \cdot \binom{6}{4} \cdot \binom{10}{2}$ different license plates are possible [4 points].

(c) [11 points] How many different license plates are possible where no letter occurs more than once and exactly three different digits occur, with these digits occurring 1 time, 2 times, and 3 times respectively (e.g. COMB 338328)? Do *not* leave any (binomial/)multinomial coefficient in the final answer.

Solution: There are $26 \cdot 25 \cdot 24 \cdot 23$ possible ways to choose the 4 distinct letters [2 point]. There are

$\binom{6}{1,2,3}(= \frac{6!}{1!\cdot 2!\cdot 3!} = \frac{6!}{2\cdot 3!})$ possible ways to choose (unordered) places for the digits occurring 1 time, 2 times, and 3 times [2 points]. There are $10\cdot 9\cdot 8$ possible ways to (first) choose a digit that occurs 1 time, (then) choose another digit that occurs 2 times, and (then) choose another digit that occurs 3 times [2 points]. So the total number of different possible codes equals $26\cdot 25\cdot 24\cdot 23\cdot \binom{6}{1,2,3}\cdot 10\cdot 9\cdot 8 = \frac{26\cdot 25\cdot 24\cdot 23\cdot 10\cdot 9\cdot 8\cdot 6!}{2\cdot 3!}$ [5 points, 1 of which for writing out the definition of the multinomial coefficient].

4. [15 points] Using mathematical induction, prove that for all $n \in \mathbb{N}$:

$$2^{2n+1} - 5\cdot 10^n \text{ is divisible by } 6.$$

Solution:

Base case: $2^3 - 5\cdot 10^1 = 8 - 50 = -42$ is divisible by 6, so the statement holds for $n = 1$ [5 points].

Inductive case: Assume that $m \geq 1$ and that $2^{2m+1} - 5\cdot 10^m$ is divisible by 6 (writing n, or any other free symbol, instead of m is of course also fine) [3 points]. We compute [4 points, namely 2 per line]:

$$\begin{aligned} 2^{2(m+1)+1} - 5\cdot 10^{m+1} &= 4\cdot 2^{2m+1} - 5\cdot 10\cdot 10^m = (10-6)\cdot 2^{2m+1} - 5\cdot 10\cdot 10^m \\ &= 10\cdot \left(2^{2m+1} - 5\cdot 10^m\right) - 6\cdot 2^{2m+1}. \end{aligned}$$

Looking at the last line, the first term is divisible by 6 by the inductive hypothesis and the second term is obviously divisible by 6, therefore the difference is divisible by 6; it follows that $2^{2(m+1)+1} - 5\cdot 10^{m+1}$ is divisible by 6, which was to be shown [3 points].

5. Let $f : [-\pi,\pi]^2 \to \mathbb{R}^2$ be the function given by

$$f(x,y) := (\cos(2x), 3\sin(y))$$

and consider the sets $A := \left(0, \frac{\pi}{2}\right) \times \left(-\frac{\pi}{2}, \pi\right)$ and $B := [-1,0) \times (0,3]$.

(a) [7 points] Determine the image of A under f.

Solution: With $f_1(x) := \cos(2x)$ we have $f_1((0,\pi/2)) = \cos((0,\pi)) = (-1,1)$ [2 points, namely 0.5 points for each boundary point and for each bracket]. With $f_2(y) := 3\sin(y)$ we have $f_2((-\pi/2,\pi)) = 3\sin((-\pi/2,\pi)) = 3\cdot(-1,1] = (-3,3]$ (note that the right boundary point is included, as $\sin((-\pi/2,\pi)) \supset \sin((-\pi/2,\pi/2]) = (-1,1]$) [3 points, namely 0.5 points for the left boundary point and for the left bracket, 1 point for the right boundary point and for the right bracket]. Since $f(x,y) = (f_1(x), f_2(y))$, we arrive at $f(A) = f_1((0,\pi/2)) \times f_2((-\pi/2,\pi)) = (-1,1) \times (-3,3]$ [2 points].

(b) [7 points] Determine the inverse image of B under f.

Solution: We use the notation $f(x,y) = (f_1(x), f_2(y))$ as in part (a). We have $f_1^{-1}([-1,0)) = (-(3/4)\pi, -(1/4)\pi) \cup ((1/4)\pi, (3/4)\pi)$ [3 points: 1.5 points for each interval with -0.5 points subtracted for each wrong boundary point/bracket with no more than 1.5 points subtracted per interval]. We have $f_2^{-1}((0,3]) = (0,\pi)$ [2 points: 0.5 for each boundary point (2 times) and for each bracket (2 times)]. Finally, we arrive at $f^{-1}(B) = f_1^{-1}([-1,0)) \times f_2^{-1}((0,3]) = ((-(3/4)\pi, -(1/4)\pi) \cup ((1/4)\pi, (3/4)\pi)) \times (0,\pi)$ (which may also be written as the union of 2 products of intervals) [2 points].