Faculty of Sciences	X_400621: Sets and Combinatorics Exam
Vrije Universiteit Amsterdam	Thursday, February 2, 2023 (18:45-21:00)

This exam consists of 5 questions and a total of 90 points can be obtained. The grade is calculated as (number of points + 10)/10. Use of a calculator is not allowed.

Please provide a valid argument for every question!

1. In this exercise we consider the set-theoretic equality

$$(A\backslash B)^c \cap (C\backslash B)^c = B \cup (A \cup C)^c.$$

(a) [7 points] Use Venn-diagrams to show that this equality holds for arbitrary sets $A, B, C \subset U$.

(b) [10 points] Now prove the above equality, either by using the algebra of sets, or by formal reasoning (your choice).

`Note: For a proof given by the algebra of sets, indicate the laws used at each step.`

2. In the universe $U := (0, 2]$ consider the sets A_k, for $k \in \mathbb{N}$, given by

$$A_k := \left[2 - \frac{1}{3^{k-1}}, 2 - \frac{1}{3^k}\right].$$

(a) [5 points] Write A_k^c as a union of two intervals.

(b) [8 points] Determine a simple explicit expression for the set $\bigcup_{k=1}^{\infty} A_k$.

(c) [4 points] Determine whether the set $\{A_1, A_2, A_3, \ldots\}$ is a partition of the set $\bigcup_{k=1}^{\infty} A_k$.

3. The code of a vault consists of eight successive digits (each being one of $0, 1, 2, \cdots, 9$).

(a) [7 point] How many different codes are possible that do not end with the digit 9?

(b) [8 points] How many different codes are possible that contain exactly two different digits, with one of these digits occurring exactly three times (so the other one occurs exactly five times, e.g. 29222929)?

(c) [11 points] How many different codes are possible that contain exactly three different digits, with the smallest of these digits occurring exactly one time and the largest of these digits occurring exactly four times (e.g. 95599259)? Write out the answer in terms of factorials (i.e. do not leave binomial/multinomial coefficients in the final answer).

Please Turn Over

4. [15 points] Using mathematical induction, prove that for all $n \in \mathbb{N}$:

$$\sum_{k=1}^{n} \left(3k^2 - 3k + 1\right) = n^3.$$

5. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the function given by

$$f(x, y) := ((x + 2)^2 - 2, y^3 + 1)$$

and consider the set $A := [-2, 2) \times (0, 2]$.

(a) [4 points] Determine the image of A under f.

(b) [5 points] Determine the inverse image of A under f.

(c) [6 points] Is f: injective? surjective? bijective?