Faculty of Sciences	X_400621: Sets and Combinatorics Exam
Vrije Universiteit Amsterdam	Thursday, February 2, 2023 (18:45-21:00)

This exam consists of 5 questions and a total of 90 points can be obtained. The grade is calculated as (number of points + 10)/10. Use of a calculator is not allowed.

Please provide a valid argument for every question!

1. In this exercise we consider the set-theoretic equality

$$(A\backslash B)^c \cap (C\backslash B)^c = B \cup (A \cup C)^c.$$

(a) [7 points] Use Venn-diagrams to show that this equality holds for arbitrary sets $A, B, C \subset U$.

Draw them. 3 point for the right hand side (1 point for each of these: $A \cup C$, $(A \cup C)^c$, $B \cup (A \cup C)^c$) and 3 points for the left hand side (1 points for each of these: $A\backslash B$ and $C\backslash B$ together, $(A\backslash B)^c$ and $(C\backslash B)^c$ together, $(A\backslash B)^c \cap (C\backslash B)^c$). 1 point for the (possibly implicit) conclusion that the sides agree.

(b) [10 points] Now prove the above equality, either by using the algebra of sets, or by formal reasoning (your choice).

`Note: For a proof given by the algebra of sets, indicate the laws used at each step.`

Solution:

- Using algebra of sets:

$$\begin{aligned}
(A\backslash B)^c \cap (C\backslash B)^c &\overset{def.\ of\ \backslash\ (\times 2)}{=} (A \cap B^c)^c \cap (C \cap B^c)^c \\
&\overset{De\ Morgan\ (\times 2)}{=} (A^c \cup (B^c)^c) \cap (C^c \cup (B^c)^c) \\
&\overset{involution\ (\times 2)}{=} (A^c \cup B) \cap (C^c \cup B) \\
&\overset{commutativity\ (\times 2)}{=} (B \cup A^c) \cap (B \cup C^c) \\
&\overset{distributivity}{=} B \cup (A^c \cap C^c) \\
&\overset{DeMorgan}{=} B \cup (A \cup C)^c
\end{aligned}$$

  Grading: 1 point for each identity used (2 points for the ($\times 2$) equalities, written out in either one or two steps). Minus points if names are missing/incorrect, disregarding the ($\times 2$), as follows: -0.5, -1, -1.5, -2 points for 1, 2–3, 4–5, 6 missing/incorrect names respectively.

- By formal reasoning: $x \in (A\backslash B)^c \cap (C\backslash B)^c \Leftrightarrow x \in (A\backslash B)^c$ and $x \in (C\backslash B)^c$
  $\Leftrightarrow$ it is not the case that $x \in A\backslash B$ and it is not the case that $x \in C\backslash B$
  $\Leftrightarrow$ it is not the case that ($x \in A$ and $x \notin B$) and it is not the case that ($x \in C$ and $x \notin B$)
  $\Leftrightarrow$ ($x \notin A$ or $x \in B$) and ($x \notin C$ or $x \in B$) $\Leftrightarrow$ ($x \in B$ or $x \notin A$) and ($x \in B$ or $x \notin C$)
  $\Leftrightarrow x \in B$ or ($x \notin A$ and $x \notin C$)
  $\Leftrightarrow x \in B$ or it is not the case that ($x \in A$ or $x \in C$) $\Leftrightarrow x \in B$ or it is not the case that $x \in A \cup C$
  $\Leftrightarrow x \in B$ or $x \in (A \cup C)^c \Leftrightarrow x \in B \cup (A \cup C)^c$

  Grading: Steps may be (somewhat) larger, and the order may be somewhat different. Proportion of 10 points, relative to the above 10 step solution.

2. In the universe $U := (0, 2]$ consider the sets A_k, for $k \in \mathbb{N}$, given by

$$A_k := \left[2 - \frac{1}{3^{k-1}}, 2 - \frac{1}{3^k}\right].$$

(a) [5 points] Write A_k^c as a union of two intervals.

Solution:

$$A_k^c = \left(0, 2 - \frac{1}{3^{k-1}}\right) \cup \left(2 - \frac{1}{3^k}, 2\right]$$

Grading: 0.5 points for each boundary point (4 times) and 0.5 points for each bracket (4 times). 1 point for taking the union of them and writing down the whole expression.

(b) [8 points] Determine a simple explicit expression for the set $\bigcup_{k=1}^{\infty} A_k$.

Solution: For every $k \in \mathbb{N}$, we have that the right endpoint of A_k equals the left endpoint of A_{k+1} as $2 - \frac{1}{3^k} = 2 - \frac{1}{3^{(k+1)-1}}$. Together with $A_1 = [1, \frac{5}{3}]$, we get that $\bigcup_{k=1}^{n} A_k$ is equal to $[1, 2 - \frac{1}{3^k}]$. Since $1/3^k$ can be made arbitrary small (by taking k sufficiently large) but cannot equal 0, it follows that $\bigcup_{k=1}^{\infty} A_k \supset [1, 2)$. Since clearly $\bigcup_{n=1}^{\infty} A_n \subset [1, 2)$ we can conclude that $\bigcup_{k=1}^{\infty} A_k = [1, 2)$.

Grading: 2 points for the correct answer (0.5 points for each boundary point and 0.5 points for each bracket). 3 points for $\bigcup_{k=1}^{n} A_k = [1, 2 - \frac{1}{3^k}]$ (including argument on endpoints). 2 points for concluding $\bigcup_{k=1}^{\infty} A_k \supset [1, 2)$. 1 point for noting $\bigcup_{n=1}^{\infty} A_n \subset [1, 2)$ and drawing the final conclusion.

(c) [4 points] Determine whether the set $\{A_1, A_2, A_3, \ldots\}$ is a partition of the set $\bigcup_{k=1}^{\infty} A_k$.

Solution: The set $\{A_1, A_2, \cdots\}$ is not a partition of the given set. Namely, the condition of the $A_k's$ being pairwise disjoint is not satisfied, e.g. $A_1 \cap A_2 = \left\{\frac{5}{6}\right\}$ (in fact, for every $k \in \mathbb{N}$, A_k and A_{k+1} are not disjoint).

Grading: 1 point for the correct answer, 2 points for mentioning that pairwise disjointness is one of the (three) conditions for being a partition, 1 point for noting that this condition is not satisfied.

3. The code of a vault consists of eight successive digits (each being one of $0, 1, 2, \cdots, 9$).

(a) [7 point] How many different codes are possible that do not end with the digit 9?

Solution: As there are (independently) 10 different values possible for each of the first 7 digits and 9 different values possible for the last digit [3 points], there are (by the rule of the product) $9 \cdot 10^7$ different codes possible [4 points].
(Alternatively, one could explain that this can be modelled by drawing balls with replacement with order from an urn containing 10 different outcomes for the first 7 digits, and 9 different outcomes for the last digit, thus with 10 different balls, where for the last draw one ball is removed; explanation worth 3 points.)

(b) [8 points] How many different codes are possible that contain exactly two different digits, with one of these digits occurring exactly three times (so the other one occurs exactly five times, e.g. 29222929)?

Solution: There are $\binom{8}{3} (= \binom{8}{5})$ possible ways to choose the 3 places (without order) for the digit occurring 3 times, fixing 5 (unordered) places for the other digit [2 points]. There are $10 \cdot 9$ different possibilities to (first) choose a digit that occurs 3 times and (then) another digit that occurs 5 times [2 points]. So a total combination of $9 \cdot 10 \cdot \binom{8}{3}$ different codes are possible [4 points].

(c) [11 points] How many different codes are possible that contain exactly three different digits, with the smallest of these digits occurring exactly one time and the largest of these digits occurring exactly four times (e.g. 95599259)? Write out the answer in terms of factorials (i.e. do not leave binomial/multinomial coefficients in the final answer).

Solution: The middle digit occurs $8 - 1 - 4 = 3$ times, so there are $\binom{8}{1,3,4}$ possible ways to choose (unordered) places for the digits occurring 1 time, 3 times, and 4 times [3 points]. Choosing a smallest, middle, and largest digit amounts to choosing an ordered triple of ascending digits, which can be done in $\binom{10}{3}$ different ways [3 points]. So the total number of different possible codes equals $\binom{8}{1,3,4} \cdot \binom{10}{3} = \frac{8!}{1! \cdot 3! \cdot 4!} \cdot \frac{10}{3! \cdot 7!} = \frac{8! \cdot 10!}{3!^2 \cdot 4! \cdot 7!}$ [5 points, 2 of which for writing out the definition of the binomial/multinomial coefficient].

4. [15 points] Using mathematical induction, prove that for all $n \in \mathbb{N}$:

$$\sum_{k=1}^{n} \left(3k^2 - 3k + 1\right) = n^3.$$

Solution:

Base case: $\sum_{k=1}^{1} \left(3k^2 - 3k + 1\right) = 3 \cdot 1^2 - 3 \cdot 1 + 1 = 1 = 1^3$, so the formula holds for $n = 1$ [5 points].

Inductive case: Assume that $m \geq 1$ and that $\sum_{k=1}^{m} \left(3k^2 - 3k + 1\right) = m^3$ (writing n, or any other free symbol, instead of m is of course also fine) [3 points]. We need to show that $\sum_{k=1}^{m+1} \left(3k^2 - 3k + 1\right) = (m+1)^3$ [2 points] (also OK if this is done implicitly, e.g. by writing the steps below). Indeed:

$$\sum_{k=1}^{m+1} \left(3k^2 - 3k + 1\right) = \sum_{k=1}^{m} \left(3k^2 - 3k + 1\right) + 3(m+1)^2 - 3(m+1) + 1 = m^3 + 3(m+1)^2 - 3(m+1) + 1$$
$$= m^3 + 3m^2 + 3m + 1 = (m+1)^3 \text{ (by Newton's binomial thm., or just expanding).}$$

First line of equations: 3 points. Last line: 2 points, namely 1 point for the equality and 1 point for any (minimalistic) explanation/calculation for it.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the function given by

$$f(x, y) := ((x+2)^2 - 2, y^3 + 1)$$

and consider the set $A := [-2, 2) \times (0, 2]$.

(a) [4 points] Determine the image of A under f.

Solution: Set $f_1(x) = (x+2)^2 - 2$ and $f_2(y) = y^3 + 1$ so that $f(x, y) = (f_1(x), f_2(y))$ and $f(A) = f_1([-2, 2)) \times f_2((0, 2])$. Since $f_1([-2, 2)) = [-2, 14)$ [1.5 points] and $f_2((0, 2]) = (1, 9]$, [1.5 points] we have $f(A) = [-2, 14) \times (1, 9]$ [1 point].
(-0.5 points for each wrong bracket/endpoint, with no more than 1.5 points subtracted per component.)

(b) [5 points] Determine the inverse image of A under f.

Solution: With notation $f(x,y) = (f_1(x), f_2(y))$ as in part (a), we have $f^{-1}(A) = f_1^{-1}([-2,2)) \times f_2^{-1}((0,2])$. Since $f_1^{-1}([-2,2)) = (-4,0)$ [2.5 points: 1 point for correct left endpoint, 0.5 points for the other endpoint and for each of the brackets] and $f_2^{-1}((0,2]) = (-1,1]$, [1.5 points, -0.5 points for each wrong bracket/endpoint, with no more than 1.5 points subtracted here of course], we have $f^{-1}(A) = (-4,0) \times (-1,1]$ [1 point].

(c) [6 points] Is f: injective? surjective? bijective?

Solution: Not injective [1 point], since e.g $f(0,0) = (2,1) = f(-4,0)$ [1 point]. Not surjective [1 point], since e.g. $(-3,0) \in \mathbb{R}^2$ is not in the range [1 point]. Not bijective [1 point], since f is not injective/since f is not surjective [1 point, stating one of these is enough].