Faculty of Sciences	Resit Sets and Combinatorics
Vrije Universiteit Amsterdam	July 09, 2020 (10:00-12:00)

This exam consists of 5 questions and a total of 27 points can be obtained. The grade is calculated as (number of points $+ 3)/3$. Use of a calculator is not allowed.

Please provide an argument at every question!

1. In this exercise we consider a set-theoretic equality involving the *symmetric difference* between two sets A and B, which is defined as $(A \backslash B) \cup (B \backslash A)$. The equality reads

$$(A \backslash B) \cup (B \backslash A) = (A \cup B) \backslash (A \cap B).$$

(a) [3 points] Use Venn-diagrams to show that this equality holds for all sets A and B. Clearly illustrate how the sets on each side of the equality are constructed, by drawing a new Venn-diagram for every set operation used in the construction.

(b) [4 points] Prove the equality by using either the algebra of sets or formal reasoning.

2. The limit of an increasing sequence of sets $A_1 \subset A_2 \subset \cdots$ is defined as $\lim_{k\to\infty} A_k = \bigcup_{k=1}^{\infty} A_k$ and the limit of a decreasing sequence of sets $A_1 \supset A_2 \supset \cdots$ is defined as $\lim_{k\to\infty} A_k = \bigcap_{k=1}^{\infty} A_k$. Let the universe $\mathcal{U} = \mathbb{R}$ and define a sequence of intervals by $A_k = (1/2, 1 + 1/k)$ for $k \in \mathbb{N}$.

(a) [1 point] Write A_k^c as the union of two intervals.

(b) [1 point] Is $A_1^c, A_2^c, A_3^c, \cdots$ an increasing or a decreasing sequence?

(c) [2 points] Determine $\lim_{k\to\infty} A_k^c$.

3. An ATM personal identification number (PIN) consists of four successive digits (each being one of $0, 1, 2, \cdots, 9$).

(a) [1 point] How many different PINs are possible if there are no restrictions on the choice of digits?

(b) [2 points] How many PINs are possible that contain exactly two different digits, both occurring exactly twice (for example 1212)?

(c) [2 points] There are in fact restrictions (at least in the US), namely the following choices are not allowed: (i) four identical digits; (ii) ascending or descending sequences of consecutive digits such as 7654; and (iii) sequences starting with 19 (birth years are too easy to guess). How many PINs are not allowed?

See other side

4. [5 points] Let $f(x) = \ln x$ (the natural logarithm of x). We claim that the n-th derivative of f is given by

$$\cancel{f^{(n)}(x) = (-1)^n \frac{(n-1)!}{x^n}},$$

for $n \in \mathbb{N}$. Prove this formula using mathematical induction.

$$\boxed{f^{(n)}(x) = (-1)^{n-1} \frac{(n-1)!}{x^n}}$$

5. We consider the function $f\colon [0,1) \to S^1$ from the interval $[0,1) \subset \mathbb{R}$ to the unit circle $S^1 = \{(x,y) \in \mathbb{R}^2 \colon x^2 + y^2 = 1\}$ defined by $f(t) = (\cos 2\pi t, \sin 2\pi t)$.

(a) [1 point] Determine the pre-image $f^{-1}(1,0)$ of the point $(1,0) \in S^1$.

(b) [2 points] Is f injective?

(c) [2 points] Is f surjective?

(d) [1 point] Do $[0,1)$ and S^1 have the same cardinality (i.e. are they equipotent)?