Faculty of Sciences	Exam Sets and Combinatorics
Vrije Universiteit Amsterdam	Thursday, January 31, 2019 (15:15-17:15)

This exam consists of 5 questions and a total of 27 points can be obtained. The grade is calculated as (number of points + 3)/3. Use of a calculator is not allowed.

Please provide an argument at every question!

1. In this exercise we consider a set-theoretic equality involving the *symmetric difference* between two sets A and B, which is defined as $(A\backslash B) \cup (B\backslash A)$. The equality reads

$$(A\backslash B) \cup (B\backslash A) = (A \cup B)\backslash(A \cap B).$$

(a) [3 points] Use Venn-diagrams to show that this equality holds for all sets A and B. Clearly illustrate how the sets on each side of the equality are constructed, by drawing a new Venn-diagram for every set operation used in the construction.

Grading: Drawing $A\backslash B$ (0.75 points), drawing $B\backslash A$ (0.75 points), drawing $A \cup B$ (0.5 points), drawing $A \cap B$ (0.5 points), and drawing either $(A\backslash B) \cup (B\backslash A)$ or $(A \cup B)\backslash(A \cap B)$ (0.5 points).

(b) [4 points] Prove the equality by using either the algebra of sets or formal reasoning.

Solution:

Using the algebra of sets:

$$\begin{aligned}
(A\backslash B) \cup (B\backslash A) &= (A \cap B^c) \cup (B \cap A^c) \\
&= ((A \cap B^c) \cup B) \cap ((A \cap B^c) \cup A^c) \\
&= ((A \cup B) \cap (B^c \cup B)) \cap ((A \cup A^c) \cap (B^c \cup A^c)) \\
&= ((A \cup B) \cap \mathcal{U}) \cap (\mathcal{U} \cap (A \cap B)^c) \\
&= (A \cup B) \cap (A \cap B)^c \\
&= (A \cup B)/(A \cap B).
\end{aligned}$$

Using formal reasoning:

(i) We first show that $(A\backslash B) \cup (B\backslash A) \subset (A \cup B)\backslash(A \cap B)$. Let $x \in (A\backslash B) \cup (B\backslash A)$. Then $x \in (A\backslash B)$ or $x \in (B\backslash A)$. If $x \in (A\backslash B)$, then $x \in A$ and if $x \in (B\backslash A)$, then $x \in B$. In any case: $x \in A \cup B$. Furthermore, $x \in B^c$ or $x \in A^c$, so that $x \notin A \cap B$, which means that $x \in (A \cap B)^c$. Hence $x \in (A \cup B) \cap (A \cap B)^c$, which means that $x \in (A \cup B)/(A \cap B)$.

(ii) We now show that $(A\backslash B) \cup (B\backslash A) \supset (A \cup B)\backslash(A \cap B)$. Let $x \in (A \cup B)\backslash(A \cap B)$, which means that $x \in A \cup B$ and $x \notin A \cap B$. If $x \in A$, then $x \notin B$ and hence $x \in A/B$. Likewise: if $x \in B$, then $x \notin A$ and hence $x \in B/A$. Since $x \in A$ or $x \in B$, it holds that $x \in (A/B) \cup (B/A)$.

Grading: Using the algebra of sets: 4 points if correct. If incorrect; 1 point for every non-trivial step (definition of set-theoretic difference, de Morgan, etc.). Using formal reasoning: 4 point if correct (2 points for (i) and 2 points for (ii)). If incorrect: 0.5 points for every non-trivial step.

2. The limit of an increasing sequence of sets $A_1 \subset A_2 \subset \cdots$ is defined as $\lim_{k\to\infty} A_k = \bigcup_{k=1}^{\infty} A_k$ and the limit of a decreasing sequence of sets $A_1 \supset A_2 \supset \cdots$ is defined as $\lim_{k\to\infty} A_k = \bigcap_{k=1}^{\infty} A_k$. Let the universe $\mathcal{U} = \mathbb{R}$ and define a sequence of intervals by $A_k = (1/2, 1+1/k)$ for $k \in \mathbb{N}$.

(a) [1 point] Write A_k^c as the union of two intervals.

Solution: $A_k^c = (-\infty, 1/2] \cup [1+1/k, \infty)$.

Grading: 0.5 points for each of the two intervals.

(b) [1 point] Is $A_1^c, A_2^c, A_3^c, \cdots$ an increasing or a decreasing sequence?

Solution: A_k^c consists of the union of two intervals of which the left interval ($(-\infty, 1/2]$) is independent of k and the right interval ($[1+1/k, \infty)$) is an increasing sequence. Therefore, A_k^c is also an increasing sequence.

Grading: 0.5 points for correct answer and 0.5 points for correct argument.

(c) [2 points] Determine $\lim_{k\to\infty} A_k^c$.

Solution:

$$\begin{aligned}
\lim_{k\to\infty} A_k^c &= \bigcup_{k=1}^{\infty} A_i^c \\
&= \bigcup_{k=1}^{\infty} ((-\infty, 1/2] \cup [1+1/k, \infty)) \\
&= (-\infty, 1/2] \cup \left(\bigcup_{k=1}^{\infty} [1+1/k, \infty) \right) \\
&= (-\infty, 1/2] \cup (1, \infty).
\end{aligned}$$

Grading: 2 points for correct answer. If incorrect: 0.5 point for every non-trivial step.

3. An ATM personal identification number (PIN) consists of four successive digits (each being one of $0, 1, 2, \cdots, 9$).

(a) [1 point] How many different PINs are possible if there are no restrictions on the choice of digits?

Solution: For each digit there are 10 choices and the digits can be chosen independently of each other, so the number of PINs is 10^4.

Grading: correct (1 point), incorrect (0 points).

(b) [2 points] How many PINs are possible that contain exactly two different digits, both occurring exactly twice (for example 1212)?

Solution: The number of possible PINs with this property equals the product of the number of ways one can choose 2 digits and the number of ways one can order (two copies of) the chosen digits. The number of ways one can choose 2 digits equals $10!/(8!2!)$ (the order is not important). After choosing the digits, we have to select their locations. There are 4 different locations so the number of possible locations for one of the two chosen digits equals $4!/(2!2!) = 6$. The locations of the second digits are then completely determined. The number of possible PINs equals 270.

Grading: 1 point for each of the two sub-arguments above. Subtract 0.5 for a calculation mistake. Comment: The students don't have to explicitly calculate the answer: something like $10!/(8!2!) \times 6$ is sufficient.

(c) [2 points] There are in fact restrictions (at least in the US), namely the following choices are not allowed: (i) four identical digits; (ii) ascending or descending sequences of consecutive digits such as 7654; and (iii) sequences starting with 19 (birth years are too easy to guess). How many PINs are not allowed?

Solution: The number of PINs that are not allowed equals the sum of the numbers of digits that satisfy (i), (ii), and (iii), which equals $10 + 2 \times 7 + 10^2 = 124$.

Grading: 0.5 points for each of the cases (i), (ii), and (iii), and 0.5 point for the correct answer.

4. [5 points] Let $f(x) = \ln x$ (the natural logarithm of x). We claim that the n-th derivative of f is given by

$$f^{(n)}(x) = (-1)^n \frac{(n-1)!}{x^n},$$

for $n \in \mathbb{N}$. Prove this formula using mathematical induction.

Solution: COMMENT: The correct formula is $f^{(n)}(x) = (-1)^{(n-1)} \frac{(n-1)!}{x^n}$.

Basecase: For $n = 1$ the formula gives $f^{(1)}(x) = x^{-1}$, which is correct.

Induction step: Suppose that the formula is correct for $n = m$. Then

$$\begin{aligned} f^{(m+1)}(x) &= \frac{d}{dx} f^{(m)}(x) \\ &= (-1)^{(m-1)} \frac{d}{dx} \frac{(m-1)!}{x^m} \\ &= -m(-1)^{(m-1)}! \frac{(m-1)!}{x^{m+1}} \\ &= (-1)^m \frac{m!}{x^{m+1}}. \end{aligned}$$

Grading: 1 point for the base case, 1 point for the first equality, 1 point for correctly calculating the derivative, and 2 points for rewriting the expression (i.e. after taking the derivative) in the desired form. The last 2 points can be subdivided into 4×0.5 points for partially correct answers.

5. We consider the function $f: [0,1) \to S^1$ from the interval $[0,1) \subset \mathbb{R}$ to the unit circle $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ defined by $f(t) = (\cos 2\pi t, \sin 2\pi t)$.

(a) [1 point] Determine the pre-image $f^{-1}(1,0)$ of the point $(1,0) \in S^1$.

Solution: The pre-image of $(1, 0)$ consists of all $t \in [0, 1)$ such that $\cos 2\pi t = 1$ and $\sin 2\pi t = 0$. This equals the intersection between $[0, 1)$ and $0 + 2\pi k$, for $k \in \mathbb{Z}$, which only contains 0. Thus, $f^{-1}(1, 0) = \{0\}$.

Grading: Correct (1 point), incorrect (0 points). Subtract 0.5 for a small error.

(b) [2 points] Is f injective?

Solution: Yes, f is injective because $f(s) = f(t)$ implies that $s = t + 2\pi k$, for some $k \in \mathbb{Z}$. Since $s, t \in [0, 1)$, this is impossible unless $s = t$.

Grading: 1 point for correct answer and 1 point for correct argument.

(c) [2 points] Is f surjective?

Solution: Yes, f is surjective because any point in S^1 can be written as $(\cos \alpha, \sin \alpha)$ for certain angle $\alpha \in [0, 2\pi)$, and the pre-image of this point is $t = \alpha/2\pi \in [0, 1)$. Thus, every point in S^1 has a pre-image.

Grading: 1 point for correct answer and 1 point for correct argument. Subtract 0.5 point for partial argument.

(d) [1 point] Do $[0, 1)$ and S^1 have the same cardinality (i.e. are they equipotent)?

Solution: Yes: The function $f\colon [0, 1) \to S^1$ is injective and surjective and hence is a bijection between $[0, 1)$ and S^1 (Schröder-Bernstein theorem). Thus, $[0, 1)$ and S^1 per definition have the same cardinality.

Grading: 0.5 point for correct answer and 0.5 point for correct argument.