Exam Sets and Combinatorics
February 1, 2018, 15:15–17:15

This exam consists of 6 assignments with a total of 11 parts, for which you can obtain a maximum possible score of 36 points. The grade is computed as (number of points + 4)/4. You are not allowed to use aids such as calculators or notes. A brief and clear explanation should accompany all your answers.

1. In this assignment we consider the set-theoretic equality

$$(A^c \cup C) \setminus (A \cap B) = (A \setminus (C \setminus B))^c.$$

(a) [3 points] Use Venn-diagrams to show that this equality holds for all sets A, B and C. To do so, illustrate clearly how the sets on each side of the equality are constructed, by showing, for each set operation used in the expression, the result of applying that operation in a new Venn diagram.

(b) [4 points] Now prove the above equality by using either the algebra of sets, or formal reasoning (your choice).

2. [3 points] Let $A_n := \left(-2^{1-n}, \dfrac{n}{n+1}\right]$. Determine the set $\bigcap_{n=1}^{\infty} A_n$.

3. [3 points] Consider the product sets A, B and C defined by

$$A := (0,3) \times [0,2) \qquad B := [2,5] \times (0,2] \qquad C := [1,4) \times [1,3).$$

Express the set $(A \cap B) \setminus C$ in a similar way as the product of two intervals.

4. An urn contains four red balls numbered 1 through 4, four green balls numbered 5 through 8, and four blue balls numbered 9 through 12. One by one and without replacement, we draw six balls from the urn, and write down the numbers on the drawn balls in the order in which they were drawn.

(a) [3 points] What is total number of possible outcomes?

(b) [3 points] How many outcomes are in the event that exactly three red balls and exactly three blue balls were drawn?

(c) [3 points] How many outcomes are in the event that exactly two balls of each colour were drawn, and for each colour, the two balls of that colour were drawn immediately after each other?

(see reverse side)

5. [5 points] Using mathematical induction, prove that for all $n \in \mathbb{N}$,

$$4\sum_{k=1}^{n} k(-1)^k = (-1)^n(2n+1) - 1.$$

6. In this assignment, we consider the function $f\colon \mathbb{N}^2 \to \mathbb{Z}$ defined by

$$f(n, m) = 2n - m.$$

(a) [3 points] Determine the image of the set $\{1, \ldots, 10\}^2$ under f.

(b) [3 points] Determine the inverse image of the set $\{0\}$ under f.

(c) [3 points] Determine whether or not the function f is

i. injective;
ii. surjective;
iii. bijective;

and in each case, briefly explain your answer (you are allowed to make use of your answers to (a) and (b)).