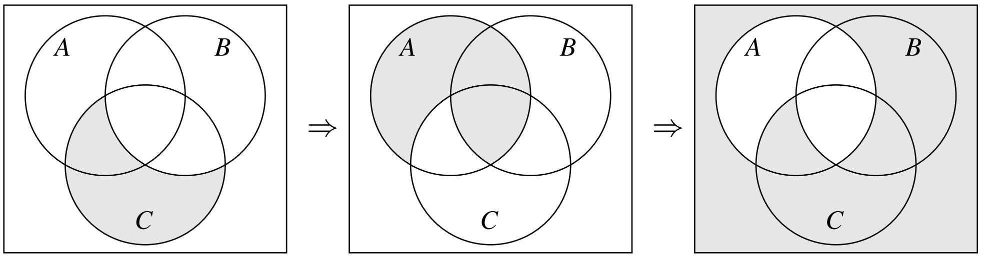
Solutions Exam Sets and Combinatorics
February 1, 2018

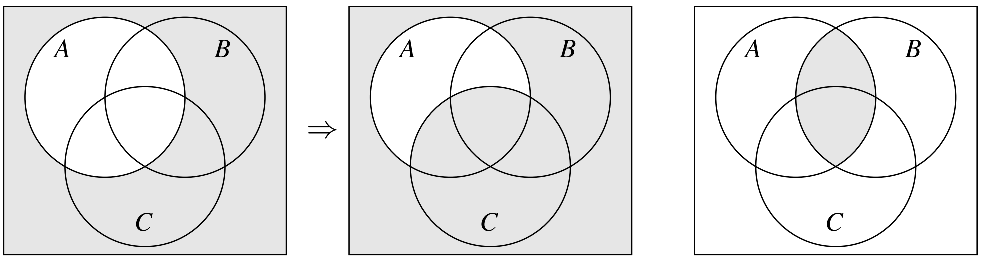
1.

(a) These Venn-diagrams show the construction of $(A \setminus (C \setminus B))^c$:



$C \setminus B$ $\quad$ $A \setminus (C \setminus B)$ $\quad$ $(A \setminus (C \setminus B))^c$

These are the Venn-diagrams for $(A^c \cup C)$ and $(A \cap B)$:



A^c $\quad$ $A^c \cup C$ $\quad$ $A \cap B$

We now see that indeed, $(A^c \cup C) \setminus (A \cap B)$ will give the same Venn-diagram as the top-right picture for $(A \setminus (C \setminus B))^c$.

(b) Using the algebra of sets, we can prove the equality as follows:

$$
\begin{aligned}
(A^c \cup C) \setminus (A \cap B) &= (A^c \cup C) \cap (A \cap B)^c && \text{(definition)} \\
&= (A^c \cup C) \cap (A^c \cup B^c) && \text{(De Morgan)} \\
&= A^c \cup (C \cap B^c) && \text{(distributivity)} \\
&= A^c \cup (C \setminus B) && \text{(definition)} \\
&= A^c \cup ((C \setminus B)^c)^c && \text{(involution)} \\
&= (A \cap (C \setminus B)^c)^c && \text{(De Morgan)} \\
&= (A \setminus (C \setminus B))^c && \text{(definition)}
\end{aligned}
$$

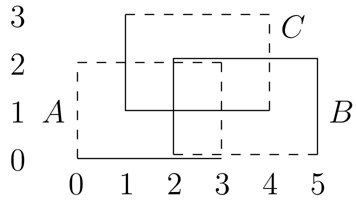
Here is a proof by logical reasoning:

Suppose that $x \in (A^c \cup C) \setminus (A \cap B)$. Then $x \in A^c \cup C$ and $x \notin A \cap B$. Hence $x \in A^c$ or $x \in C$, and it is not the case that x is in both A and B. Therefore, $x \in A^c$ or $x \in C$, and at the same time, $x \in A^c$ or $x \in B^c$. But this means that either $x \in A^c$, or else x must be both an element of C and of B^c. So $x \in A^c$, or else $x \in C \setminus B$, which means that it cannot be the case that x is both in the set A and not in the set $C \setminus B$. So it is not the case that $x \in A \setminus (C \setminus B)$, and therefore $x \in (A \setminus (C \setminus B))^c$.

This argument can be reversed: if $x \in (A \setminus (C \setminus B))^c$, then it is not the case that $x \in A \setminus (C \setminus B)$. In other words, it is not the case that $x \in A$ and $x \notin C \setminus B$. But this means that either $x \in A^c$, or else $x \in C \setminus B$. Hence, either $x \in A^c$, or else x is both an element of C and of B^c. From this it follows that $x \in A^c$ or $x \in C$, and at the same time, $x \in A^c$ or $x \in B^c$. Thus, $x \in A^c \cup C$, and it is not the case that $x \in A$ and $x \in B$. Therefore, $x \in A^c \cup C$ and $x \notin A \cap B$, which means that $x \in (A^c \cup C) \setminus (A \cap B)$.

2. The right endpoint of the interval A_n increases with n from $1/2$ to 1. The left endpoint also increases with n, from -1 to 0. So only the real numbers in $(-1, 1)$ lie in at least one of the sets A_n, and therefore $\bigcap_{n=1}^{\infty} A_n \subset (-1, 1)$. But for $x > 1/2$ we have that $x \notin A_1$, so these points are not in $\bigcap_{n=1}^{\infty} A_n$. And for $x < 0$, we have that $x \notin A_n$ if we choose n large enough, so these points are also not in $\bigcap_{n=1}^{\infty} A_n$. Since every point in the interval $[0, 1/2]$ does lie in A_n for every $n \in \mathbb{N}$, we conclude that $\bigcap_{n=1}^{\infty} A_n = [0, 1/2]$.

3. We first draw a picture of the sets A, B and C in $\mathbb{R}^2$. Each of these sets is a rectangle. We use solid lines for sides of rectangles that *are* part of the set, and dashed lines for sides that *are not* part of the set:



As we see in the picture, $A \cap B$ is equal to the product set $[2, 3) \times (0, 2)$. If we then remove the points in C, we get the set

$$(A \cap B) \setminus C = [2, 3) \times (0, 1).$$

4.

(a) The total number of outcomes is $12!/(12-6)! = 12!/6! = 12{\cdot}11{\cdot}10{\cdot}9{\cdot}8{\cdot}7$.

(b) First, choose the three draws that yield a blue ball. This fixes the positions in the outcome that correspond to the blue balls, and the positions that correspond to the red balls. We then still need to choose the three numbers of the blue balls (with order), and the three numbers of the red balls (with order). Hence the desired number of outcomes is:

$$\binom{6}{3} \cdot (4 \cdot 3 \cdot 2) \cdot (4 \cdot 3 \cdot 2) = \frac{6!}{3!\,3!} \cdot 4! \cdot 4! = 6! \cdot 4 \cdot 4.$$

Alternatively, we can first choose the three numbers of the red balls that are drawn, and the three numbers of the blue balls that are drawn. Given these numbers, we still have to multiply by the number of ways in which the numbers can be ordered. This gives again

$$\binom{4}{3} \cdot \binom{4}{3} \cdot 6! = \frac{4!}{3!\,1!} \cdot \frac{4!}{3!\,1!} \cdot 6! = 6! \cdot 4 \cdot 4.$$

(c) First, choose one of the 3! possible orders in which the three colours appear. Then, choose the first and second red number that are drawn, the first and second blue number, and finally, the first and second green number:

$$3! \cdot (4 \cdot 3) \cdot (4 \cdot 3) \cdot (4 \cdot 3).$$

Alternatively, first choose for each colour the numbers of the balls that are drawn of that colour. Then multiply by the number of ways in which these numbers can be arranged to give an outcome satisfying the description (3! ways to order the colours, 2 ways to order the numbers of each colour):

$$\binom{4}{2} \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot 3! \cdot 2 \cdot 2 \cdot 2 = \frac{4!\,2}{2!\,2!} \cdot \frac{4!\,2}{2!\,2!} \cdot \frac{4!\,2}{2!\,2!} \cdot 3! = 3! \cdot (4 \cdot 3) \cdot (4 \cdot 3) \cdot (4 \cdot 3).$$

5. We have to prove using mathematical induction that for all $n \in \mathbb{N}$,

$$4 \sum_{k=1}^{n} k(-1)^k = (-1)^n (2n+1) - 1. \qquad (*)$$

We start with the base case. For $n = 1$ we have on the one hand,

$$4 \sum_{k=1}^{n} k(-1)^k = 4 \sum_{k=1}^{1} k(-1)^k = 4 \cdot 1(-1)^1 = -4,$$

and on the other hand,

$$(-1)^n(2n+1)-1=(-1)^1(2\cdot 1+1)-1=-3-1=-4.$$

This shows that $(*)$ is indeed true for $n=1$.

Next, let $m \in \mathbb{N}$ be arbitrary, and assume that

$$4\sum_{k=1}^{m} k(-1)^k = (-1)^m(2m+1)-1. \qquad \text{IH}$$

It now follows that

$$\begin{aligned}
4\sum_{k=1}^{m+1} k(-1)^k &= 4\sum_{k=1}^{m} k(-1)^k + 4(m+1)(-1)^{m+1} \\
&\overset{\text{IH}}{=} (-1)^m(2m+1)-1+4(m+1)(-1)^{m+1} \\
&= (-1)^{m+1}\left[-(2m+1)+4(m+1)\right]-1 \\
&= (-1)^{m+1}(2m+3)-1 = (-1)^{m+1}(2(m+1)+1)-1.
\end{aligned}$$

This shows that if $(*)$ is true for $n=m$, then it is also true for $n=m+1$, which completes the inductive proof.

6.

(a) The smallest possible image of the points in the set $\{1,\ldots,10\}^2$ is clearly $f(1,10)=-8$, and the largest is $f(10,1)=19$. It is not difficult to see that all intermediate integer values are attained as well. Hence

$$f(\{1,\ldots,10\}^2)=\{-8,\ldots,19\}.$$

(b) We are looking for all points $(n,m)\in\mathbb{N}^2$ for which it is the case that $f(n,m)=2n-m=0$. So this means that we must have $m=2n$, where n ranges over the natural numbers. In other words,

$$f^{-1}(\{0\})=\{(n,2n)\colon n\in\mathbb{N}\}=\{(1,2),(2,4),(3,6),\ldots\}.$$

(c) The function f is not injective, because $f(1,2)=f(2,4)=0$ (see the answer in (b)). So f cannot be bijective either. However, f is surjective, because for any number $k\in\mathbb{Z}$ it is the case that $f(n,m)=2n-m=k$ has a solution with $n,m\in\mathbb{N}$: for k positive we can take for instance $n=m=k$, and for k nonpositive we can take $n=1$ and $m=2-k$.