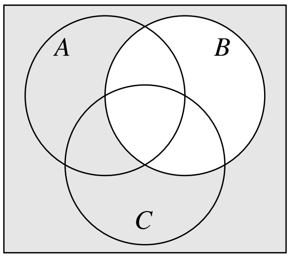
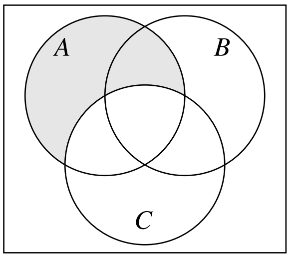
Solutions Resit Exam Sets and Combinatorics
April 3, 2017

1.

(a) These Venn-diagrams illustrate how the set $B^c \cup (A \setminus C)$ is constructed:

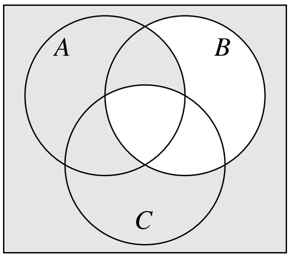


B^c



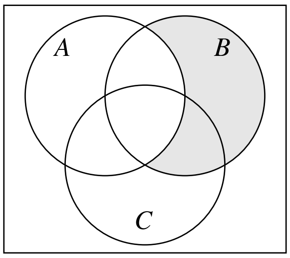
$A \setminus C$

$\Rightarrow$



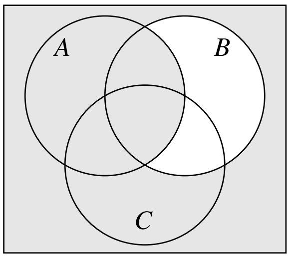
$B^c \cup (A \setminus C)$

And this shows the construction of $(B \setminus A)^c \cap (B \cap C)^c$:

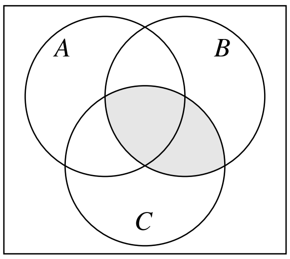


$B \setminus A$

$\Rightarrow$

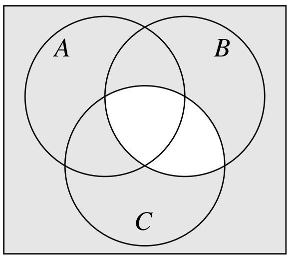


$(B \setminus A)^c$

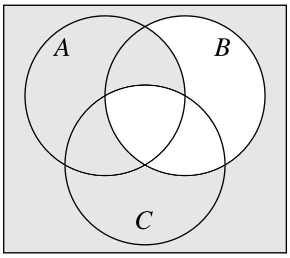


$B \cap C$

$\Rightarrow$



$(B \cap C)^c$



$(B \setminus A)^c \cap (B \cap C)^c$

Clearly, $B^c \cup (A \setminus C)$ is the same set as $(B \setminus A)^c \cap (B \cap C)^c$.

(b) We now prove this equality using the algebra of sets:

$$\begin{aligned}
B^c \cup (A \setminus C) &= B^c \cup (A \cap C^c) && \text{(definition)}\\
&= (B^c \cup A) \cap (B^c \cup C^c) && \text{(distributivity)}\\
&= (B^c \cup (A^c)^c) \cap (B^c \cup C^c) && \text{(involution)}\\
&= (B \cap A^c)^c \cap (B \cap C)^c && \text{(De Morgan)}\\
&= (B \setminus A)^c \cap (B \cap C)^c && \text{(definition)}
\end{aligned}$$

Here is a proof using formal reasoning (iff stands for "if and only if"):

$x \in B^c \cup (A \setminus C)$
iff $x \in B^c$, or $x \in A \setminus C$
iff $x \notin B$, or $x \in A$ and $x \notin C$
iff $x \notin B$ or $x \in A$, and $x \notin B$ or $c \notin C$
iff it is not the case that $x \in B$ and $x \notin A$, and it is not the case that $x \in B$ and $x \in C$
iff it is not the case that $x \in B \setminus A$, and it is not the case that $x \in B \cap C$
iff $x \in (B \setminus A)^c$ and $x \in (B \cap C)^c$
iff $x \in (B \setminus A)^c \cap (B \cap C)^c$.

2.

(a) For every $x \geq 1$ we have that $x \notin A_1$, and for every $x < 0$, it is possible to choose $n \in \mathbb{N}$ so large that $x < -\frac{1}{n}$, and hence $x \notin A_n$. Therefore, no real number in $[1, \infty)$ or in $(-\infty, 0)$ can be an element of $\bigcap_{n=1}^{\infty} A_n$. But every real number in $[0, 1)$ is an element of A_n for every $n \in \mathbb{N}$, because $-\frac{1}{n} < 0$ and $n \geq 1$ for every $n \in \mathbb{N}$. Therefore,

$$\bigcap_{n=1}^{\infty} A_n = [0, 1)$$

(b) Consider the union of the A_n: $\bigcup_{n=1}^{\infty} A_n$. Since $A_1 = [-1, 1)$, every real number in $[-1, 1)$ is an element of this union. Moreover, since $[0, n) \subset A_n$ for every $n \in \mathbb{N}$, every positive real number is an element of this union. However, since $-\frac{1}{n} \geq -1$ for every $n \in \mathbb{N}$, no real number smaller than -1 is in the union. Hence $\bigcup_{n=1}^{\infty} A_n = [-1, \infty)$ and therefore (by De Morgan)

$$\bigcap_{n=1}^{\infty} A_n^c = \Big(\bigcup_{n=1}^{\infty} A_n \Big)^c = (-\infty, -1)$$

3.

(a) This is a draw without order and without replacement of 4 balls from a total of 15 balls, hence the total number of outcomes is

$$\binom{15}{4} = 1\,365.$$

(b) We must choose two of the five green balls, and two of the ten non-green balls. Given these two choices, the outcome is completely determined. Hence the number of possibilities is

$$\binom{5}{2} \cdot \binom{10}{2} = 10 \cdot 45 = 450.$$

(c) Since the balls are in the order red / green / blue, there are three ways in which the second and third drawn ball can be both not green: either (i) *no* green ball was drawn at all (all drawn balls were red or blue), or (ii) three red balls and one green ball were drawn, or (iii) one green ball and three blue balls were drawn. We must sum over the number of outcomes for each case:

$$\binom{10}{4} + \binom{5}{3} \cdot \binom{5}{1} + \binom{5}{1} \cdot \binom{5}{3} = 210 + 10 \cdot 5 + 5 \cdot 10 = 310.$$

4. Our goal is to prove using mathematical induction that for all $n \in \mathbb{N}$,

$$1 + \sum_{k=1}^{n} k \cdot (k!) = (n+1)!$$

Base case: $1 + \sum_{k=1}^{1} k \cdot (k!) = 1 + 1 \cdot (1!) = 1 + 1 = 2 = 2! = (1+1)!$
Inductive step: take $n = m \in \mathbb{N}$ arbitrary and assume that

$$\textbf{(IH)} \qquad 1 + \sum_{k=1}^{m} k \cdot (k!) = (m+1)!$$

Now consider the case $n = m+1$. Using calculus and the inductive hypothesis (IH), we obtain

$$\begin{aligned} 1 + \sum_{k=1}^{m+1} k \cdot (k!) &= 1 + \sum_{k=1}^{m} k \cdot (k!) + (m+1) \cdot ((m+1)!) \\ &\overset{\text{(IH)}}{=} (m+1)! + (m+1) \cdot (m+1)! \\ &= (m+2) \cdot (m+1)! \\ &= (m+2)! \end{aligned}$$

Since $(m+2)! = ((m+1)+1)!$, this completes the inductive proof.

5. To get a feeling for how the function f behaves, try to fill in a few positive and negative numbers in the formula $f(n) = n + |n|$. Then one easily sees that f maps negative values of n to 0, and positive values of n to $2n$ (0 is mapped to 0). We can use this to answer the three questions.

(a) The set E is the set of nonnegative multiples of 2. By our observation above, these numbers are mapped to the nonnegative multiples of 4:

$$f(E) = \{4n\colon n \in \mathbb{N}_0\} = \{0, 4, 8, 12, \ldots\}.$$

(b) Since 0 and all negative integers are mapped to 0, the function is not injective. For example,

$$f(0) = f(-1) = 0.$$

(c) By our observation above, the range of the function f consists of all positive multiples of 2 and the number 0. Hence the range of f is the set E, and the function is indeed surjective. To be concrete, if we pick a specific number $2k$ in E (with $k \in \mathbb{N}_0$), then the equation $f(n) = 2k$ has a solution for $n \in \mathbb{Z}$: if we take $n = k$, then $f(n) = f(k) = k + |k| = 2k$, as required.