Exam Sets and Combinatorics
February 2, 2017, 15:15–17:15

This exam consists of 5 exercises with a total of 11 parts, for which you can obtain a maximum possible score of 36 points. The grade is computed as (number of points + 4)/4. You are not allowed to use aids such as calculators or notes. A brief and clear explanation should accompany all your answers.

1. In this exercise we consider the set-theoretic equality

$$(C^c \setminus B) \cup A = ((B \setminus A) \cup (C \setminus A))^c.$$

(a) [3 points] Use Venn-diagrams to show that this equality holds for all sets A, B and C. Clearly illustrate how the sets on each side of the equality are constructed, by drawing a new Venn-diagram for every set operation used in the construction.

(b) [4 points] Now prove the above equality by using either the algebra of sets, or formal reasoning (your choice).

2. Let A, B, C and D be the intervals of the real line defined by $A := (0, 2)$, $B := [0, 2]$, $C := [1, 3)$ and $D := (1, 3]$. Each of the two sets below can be expressed in a simpler way as a product of two intervals of $\mathbb{R}$. Determine this simpler explicit expression for the given sets:

(a) [3 points] $(A \times B) \cap (C \times D)$.

(b) [3 points] $(A \times A) \setminus ((B \times D) \cup (C \times B))$.

3. Five women and four men (nine persons in total) are seated one by one in a row. How many different ways of seating the nine persons are possible if

(a) [3 points] there are no restrictions on the seating arrangement?

(b) [3 points] the five women must sit next to one another (that is, no man is seated in between two women)?

(c) [3 points] no two men or two women can sit next to each other?

4. [5 points] Using mathematical induction, prove that for all $n \in \mathbb{N}$,

$$8 \cdot 3^{2n-1} + 2^{n+1} \qquad \text{is divisible by 7.}$$

(see reverse side)

5. Let A be the set defined by $A := \{(m, n) \in \mathbb{N}^2 \colon n > m\}$, and let $f \colon A \to \mathbb{N}$ be the function given by

$$f(m, n) := m \cdot n.$$

(a) [3 points] Determine the inverse image of the set $\{2, 4, 6, 8\}$ under f.

(b) [3 points] Determine whether the function f is injective.

(c) [3 points] Determine whether the function f is surjective.

Do not forget to briefly explain your answers!