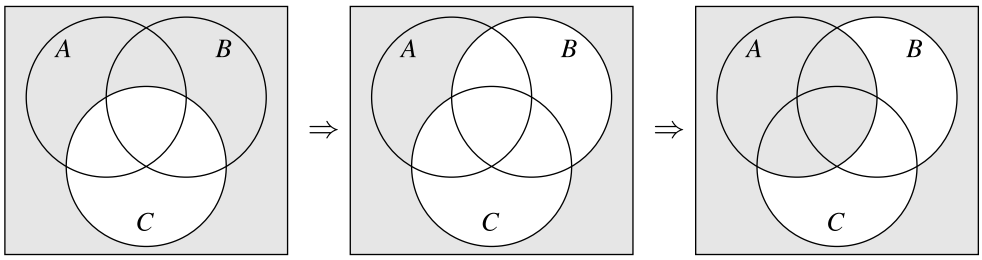
Solutions Exam Sets and Combinatorics
February 2, 2017

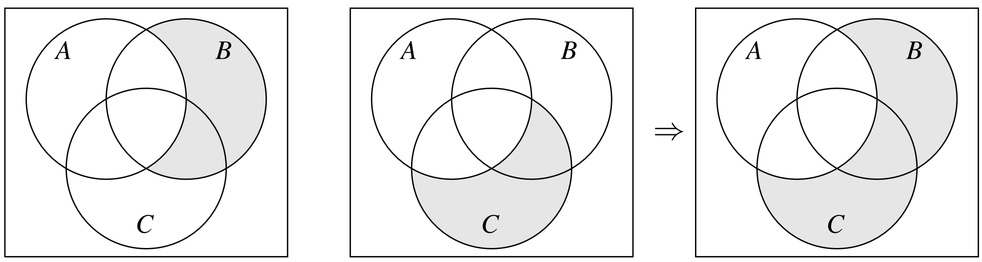
1.

(a) These Venn-diagrams illustrate how the set $(C^c \setminus B) \cup A$ is constructed:



C^c $\quad$ $C^c \setminus B$ $\quad$ $(C^c \setminus B) \cup A$

And this shows the construction of $(B \setminus A) \cup (C \setminus A)$:



$B \setminus A$ $\quad$ $C \setminus A$ $\quad$ $(B \setminus A) \cup (C \setminus A)$

Clearly, $(C^c \setminus B) \cup A$ is exactly the complement of $(B \setminus A) \cup (C \setminus A)$.

(b) We now prove this equality using the algebra of sets:

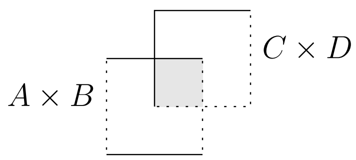
$$
\begin{aligned}
(C^c \setminus B) \cup A &= (C^c \cap B^c) \cup A && \text{(definition)} \\
&= A \cup (C^c \cap B^c) && \text{(commutativity)} \\
&= A \cup (C \cup B)^c && \text{(De Morgan)} \\
&= (A^c)^c \cup (C \cup B)^c && \text{(involution)} \\
&= (A^c \cap (C \cup B))^c && \text{(De Morgan)} \\
&= ((A^c \cap C) \cup (A^c \cap B))^c && \text{(distributivity)} \\
&= ((B \cap A^c) \cup (C \cap A^c))^c && \text{(commutativity)} \\
&= ((B \setminus A) \cup (C \setminus A))^c && \text{(definition)}
\end{aligned}
$$

Here is a proof using formal reasoning (iff stands for "if and only if"):

$$
\begin{aligned}
&x \text{ is in } (C^c \setminus B) \cup A \\
&\quad \text{iff} \quad \text{either } x \text{ is in } A, \text{ or } x \text{ is in } C^c \setminus B \\
&\quad \text{iff} \quad \text{either } x \text{ is in } A, \text{ or } x \text{ is not in } C \text{ and not in } B \\
&\quad \text{iff} \quad x \text{ is not in } C \setminus A \text{ and } x \text{ is not in } B \setminus A \\
&\quad \text{iff} \quad \text{it is not the case that } x \text{ is in } B \setminus A \text{ or in } C \setminus A \\
&\quad \text{iff} \quad x \text{ is not in } (B \setminus A) \cup (C \setminus A) \\
&\quad \text{iff} \quad x \text{ is in } ((B \setminus A) \cup (C \setminus A))^c.
\end{aligned}
$$

2.

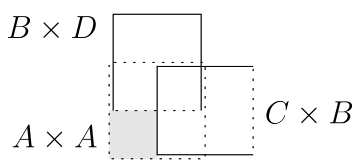
(a) It helps to draw a picture. The sets $A \times B$ and $C \times D$ are squares in $\mathbb{R}^2$. In the picture below, the points that lie on a solid boundary *are* elements of the corresponding product set, while points that lie on a dotted boundary are *not* in the set:



The intersection of the two product sets is the grey square; we conclude from the picture that

$$(A \times B) \cap (C \times D) = [1, 2) \times (1, 2].$$

(b) We again draw a picture, with the boundaries of the three squares $A \times A$, $B \times D$ and $C \times B$ drawn slightly apart so we can see clearly which points on the boundaries are and are not elements of the respective sets:



The grey area depicts the region we are after, but we must be careful: since we are *taking away* the union of $B \times D$ and $C \times B$ from $A \times A$, points on the top boundary of the grey square are *not* removed, while points on the right boundary *are* removed. Hence we conclude that

$$(A \times A) \setminus ((B \times D) \cup (C \times B)) = (0, 1) \times (0, 1].$$

3.

(a) There are 9! ways to seat nine persons in a row (this is just the number of permutations of an ordered sequence of nine different objects).

(b) One way to approach this goes as follows: if the five women are to sit together, we will have n men, followed by five women, followed by $4-n$ men as our final arrangement, where n can be any number in $\{0, 1, \ldots, 4\}$. For each n, there are 4! ways to seat the four men, and 5! ways to seat the five women, so the total number of arrangements is

$$5 \times 4! \times 5! = 14\,400.$$

Alternatively, we can treat the five women as one "entity", and each of the four men as a separate entity. Then there are 5! ways to order these five different entities, and the five women can also be ordered in 5! ways (within their entity). This again gives a total number of arrangements equal to

$$5! \times 5! = 14\,400.$$

(c) If no two men or two women can sit next to each other, the sexes must alternate, and since there are five women and four men, this implies that for every position in the row, the sex is determined. Thus, we can only choose the order among the four men, and the order among the five women, so the total number of possible arrangements is

$$4! \times 5! = 2\,880.$$

4. Our goal is to prove using mathematical induction that for all $n \in \mathbb{N}$,

$$8 \cdot 3^{2n-1} + 2^{n+1} \quad \text{is divisible by 7.}$$

Base case: $8 \cdot 3^{2\cdot 1-1} + 2^{1+1} = 8 \cdot 3^1 + 2^2 = 24 + 4 = 28$, which is indeed divisible by 7.

Inductive step: take $n = m \in \mathbb{N}$ arbitrary and assume

$$\textbf{(IH)} \qquad 8 \cdot 3^{2m-1} + 2^{m+1} \quad \text{is divisible by 7.}$$

Now consider the formula for $n = m + 1$. We can rewrite this as follows:

$$\begin{aligned} 8 \cdot 3^{2(m+1)-1} + 2^{(m+1)+1} &= 8 \cdot 3^{2m+1} + 2^{m+2} \\ &= 8 \cdot 3^2 \cdot 3^{2m-1} + 2 \cdot 2^{m+1} \\ &= 9 \cdot 8 \cdot 3^{2m-1} + 2 \cdot 2^{m+1} \\ &= 2 \cdot (8 \cdot 3^{2m-1} + 2^{m+1}) + 7 \cdot 8 \cdot 3^{2m-1} \end{aligned}$$

Both terms on the last line are divisible by 7: the first by the inductive hypothesis (IH), and the second because it is a multiple of 7. Therefore, the whole expression is divisible by 7, and this completes the proof.

5.

(a) We need to write each of the numbers 2, 4, 6 and 8 as a product of two natural numbers, with the second number being strictly larger than the first. For the numbers 2 and 4, there is only one option: $2 = 1 \cdot 2$, $4 = 1 \cdot 4$, for the numbers 6 and 8 there are two options: $6 = 1 \cdot 6 = 2 \cdot 3$, $8 = 1 \cdot 8 = 2 \cdot 4$. Therefore,

$$f^{-1}(\{2,4,6,8\}) = \{(1,2),(1,4),(1,6),(1,8),(2,3),(2,4)\}.$$

(b) The function f is not injective, because (for example) the two points $(1,8)$ and $(2,4)$ are both elements of A, and are mapped to the same image:

$$f(1,8) = f(2,4) = 8,$$

(c) The function f is not surjective either, because $f(m,n) = m \cdot n = 1$ has only one solution with $m, n \in \mathbb{N}$, namely $m = n = 1$. However, the point $(1,1)$ is not an element of A, so there is no element of A that is mapped to the number 1 by the function f.