Faculty of Science	Rings and fields (X_400630), part 2
Vrije Universiteit Amsterdam	Partial examination 21-12-2022 (15:30-17:45)

- **Attempt all problems.**
- Answers without reasoning score poorly, so give proper justifications everywhere.
- In case you cannot do a part of a problem, you may still use its stated result in the remainder of the problem.
- Calculators, notes, books, etc., may not be used.
- Do not hand in scrap, etc., and when handing in $n \geq 1$ sheets, number them $1/n, \ldots, n/n$.

(1) Factorise $11 - 3i$ into irreducibles in $\mathbb{Z}[i] = \{a + bi$ with a and b in $\mathbb{Z}\}$, the Gaussian integers.

(2) Let $R = \mathbb{Z}[\sqrt{-7}]$, a subring of $\mathbb{C}$ that is an integral domain with units $R^* = \{\pm 1\}$.
  (a) Show that 3 is a prime element of R.
  (b) Show that $1 + \sqrt{-7}$ is an irreducible element of R but *not* a prime element of R.

(3) Let x be a variable. Factorise $2x^6 - 2x^5 - 4x + 4$ into irreducibles in $\mathbb{Z}[x]$.

(4) Show that $y^2x^6 + 1$ is irreducible in $\mathbb{R}[x, y]$ **but** reducible in $\mathbb{C}[x, y]$. **Explain, as always, your method carefully, and indicate which conditions you verify before applying a theorem.**

(5) Let $R = D^{-1}\mathbb{Z}$ be the ring of fractions for $D = \{1 + 5k$ with k in $\mathbb{Z}\}$. It is given that R is an integral domain.
  (a) Show that $R^* = \{\frac{a}{d}$ in R with a not divisible by $5\}$.
  (b) Prove that $\frac{5}{1}$ is a prime element of R.
  (c) Now show that every irreducible element of R is associate to $\frac{5}{1}$.

(6) In $\mathbb{C}$, we let a be the unique positive real root of $x^8 - 2$. We also fix a root b of $x^4 + 1$.
  (a) Show that $[\mathbb{Q}(a) : \mathbb{Q}] = 8$.
  (b) Show that $\mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(b)$ and $[\mathbb{Q}(b) : \mathbb{Q}(\sqrt{2})] = 2$. *Hint: compute* $(b + b^{-1})^2$.
  (c) Now determine $[\mathbb{Q}(a, b) : \mathbb{Q}]$.

(7) It is given that $f(x) = x^2 + 4x + 5$ is irreducible in $\mathbb{F}_7[x]$, so that $\mathbb{F}_7[x]/(f(x))$ is a field F with 49 elements. With a the class of x, we have

$$F = \{b_0 + b_1 a \text{ with } b_0 \text{ and } b_1 \text{ in } \mathbb{F}_7\}.$$

  (a) Determine a formula for $\mathrm{Fr}_7(b_0 + b_1 a)$ of the shape $b_0' + b_1' a$ with b_0' and b_1' in $\mathbb{F}_7$, where Fr_7 is the Frobenius homomorphism in characteristic 7.
  (b) It is given that $E = \mathbb{F}_7[y]/(y^2 + 2)$ is also a field with 49 elements. Find an explicit field isomorphism $\psi : F \to E$, and **briefly explain** why your ψ does the job. *Hint: write elements of E in the form $d_0 + d_1 c$ with c the class of y.*

Distribution of points						
1: 8	2a: 5 2b: 10	3: 9	4: 12	5a: 6 5b: 5 5c: 4	6a: 6 6b: 7 6c: 5	7a: 6 7b: 7
Maximum total = 90						
Exam grade = 1 + Total/10						