Faculty of Science	**Rings and fields (X_400630), part 1**
Vrije Universiteit Amsterdam	**Partial examination 26-10-2022 (15:30-17:45)**

- **Attempt all problems.**
- Answers without reasoning score poorly, so give proper justifications everywhere.
- In case you cannot do a part of a problem, you may still use its stated result in the remainder of the problem.
- Calculators, notes, books, etc., may not be used.
- Do not hand in scrap, etc., and when handing in $n \geq 1$ sheets, number them $1/n, \ldots, n/n$.

(1) Let R be a commutative ring with 1 and $S = M_2(R)$, the ring of 2×2-matrices with coefficients in R. Show that the centre $Z(S) = \{X \text{ in } S \text{ with } XY = YX \text{ for all } Y \text{ in } S\}$ of S is given by

$$Z(S) = \left\{ \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \text{ with } r \text{ in } R \right\} .$$

Hint: use simple matrices in S to obtain conditions on an element of $Z(S)$.

(2) Let $R = \mathbb{Z}[i] = \{a + bi \text{ with } a \text{ and } b \text{ in } \mathbb{Z}\}$, a subring of $\mathbb{C}$. Use the extended Euclidean algorithm to determine a greatest common divisor d of $\alpha = 11 + 2i$ and $\beta = 1 - 8i$, and to write d in the form $x\alpha + y\beta$ with x and y in R.

(3) **All parts of this problem are independent of each other.**
Let $R = \mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \text{ with } a \text{ and } b \text{ in } \mathbb{Z}\}$, a subring of $\mathbb{C}$.

(a) Show that the ideal $(6, 7 - \sqrt{-5})$ of R is principal.

(b) Let S be the ring of fractions $D^{-1}R$ for $D = \{1, 3, 9, \ldots\} = \{3^m \text{ with } m \geq 0\}$. It is given that S has an identity 1_S. Show $S^* = \{\frac{\alpha}{3^m} \text{ with } m \geq 0 \text{ and } \mathrm{Nm}(\alpha) \text{ in } D\}$.

(c) Show that $\alpha = 2 - \sqrt{-5}$ and $\beta = 3$ have greatest common divisor 1 in R.

(d) Prove that the ideal $(2 - \sqrt{-5}, 3)$ of R is not all of R.

(4) **In this problem, formulate explicitly the results/theorems/... you use.**
Let $R = \mathbb{Z}[\sqrt{-7}] = \{a + b\sqrt{-7} \text{ with } a \text{ and } b \text{ in } \mathbb{Z}\}$, which is a subring of $\mathbb{C}$, and I the ideal $(2 + \sqrt{-7})$ of R.

(a) Prove that $\varphi : R \to \mathbb{Z}/11\mathbb{Z}$, given by $\varphi(a + b\sqrt{-7}) = \overline{a - 2b}$, is a ring homomorphism with kernel I.

(b) Show that there is a ring isomorphism $R/I \simeq \mathbb{Z}/11\mathbb{Z}$.

(c) Is I a maximal ideal of R? Is it a prime ideal of R?

(5) Let R be the polynomial ring $\mathbb{Q}[x]$. In R, we consider its ideals $I = (x^2 + x + 1)$, $J = (x - 1)$ and $K = (x^3 - 1)$.

(a) Show that there exists a ring isomorphism $R/K \simeq R/I \times R/J$.

(b) Determine $f(x)$ in R with $\deg(f(x)) < 3$ such that $f(x) + K$ is mapped to $(-x - 1 + I, 4 + J)$ under your map in (a).

(6) Let R be a commutative ring with 1, and I, J and K ideals of R. Show that if $R = I + J = I + K = J + K$, then $R = IJ + IK + JK$. *Hint: use $1 = 1^3$.*

Distribution of points					
1: 7	2: 8	3a: 7	4a: 8	5a: 6	6: 7
		3b: 9	4b: 7	5b: 9	
		3c: 8	4c: 7		
		3d: 7			
Maximum total = 90					
Exam grade = 1 + Total/10					