Faculty of Science	**Rings and fields (X_400630), part 2**
Vrije Universiteit Amsterdam	**Partial examination 23-12-2021 (12:15-14:30)**

- **Attempt all problems.**
- Answers without reasoning score poorly, so give proper justifications everywhere.
- In case you cannot do a part of a problem, you may still use its stated result in the remainder of the problem.
- Calculators, notes, books, etc., may not be used.
- Do not hand in scrap, etc., and when handing in $n \geq 1$ sheets, number them $1/n, \ldots, n/n$.

(1) Factorise $11 + 7i$ into irreducibles in $\mathbb{Z}[i] = \{a + bi \text{ with } a \text{ and } b \text{ in } \mathbb{Z}\}$, the Gaussian integers.

(2) Let $R = \mathbb{Z}[\sqrt{5}]$, a subring of $\mathbb{R}$ that is an integral domain.
 (a) Show that 3 is a prime element of R.
 (b) Show that $1 + \sqrt{5}$ is an irreducible element of R but *not* a prime element of R.

(3) Let $R = \mathbb{Z}[x]$, where x is a variable. Factorise $5x^4 + 5x^2 + 10$ into irreducibles in R.

(4) Show that $x^7y^2 + x^2y^3 + x^2y + y + 1$ is irreducible in $\mathbb{C}[x, y]$. **Explain, as always, your method carefully, and indicate which conditions you verify before applying a theorem.**

(5) Let $R = D^{-1}\mathbb{Z}$ be the ring of fractions for $D = \{1, 2, 4, 8, \ldots\} = \{2^m \text{ with } m \geq 0\}$. It is given that R is an integral domain.
 (a) Show that $R^* = \{\frac{\varepsilon 2^n}{2^m} \text{ with } m, n \geq 0 \text{ and } \varepsilon = \pm 1\}$.
 (b) Prove that if p is an *odd prime number* then $\frac{p}{1}$ is a prime element of R.
 (c) Now show that every irreducible element of R is associate to one of the prime elements in part (b).

(6) Let $f(x) = x^4 - 12$ in $\mathbb{Q}[x]$, a_1, a_2, a_3, a_4 its roots in $\mathbb{C}$, and $F = \mathbb{Q}(a_1, a_2, a_3, a_4) \subseteq \mathbb{C}$. Let a be the unique positive root in $\mathbb{R}$ of $f(x)$ and set $K = \mathbb{Q}(a) \subseteq F$.
 (a) Prove that $[K : \mathbb{Q}] = 4$.
 (b) Show that $[F : \mathbb{Q}] = 8$.
 (c) Determine the minimal polynomial of a over $\mathbb{Q}(b)$, where $b = ia$ with $i^2 = -1$.

(7) It is given that $f(x) = x^2 + 6x + 6$ is irreducible in $\mathbb{F}_7[x]$, so that $\mathbb{F}_7[x]/(f(x))$ is a field F with 49 elements. With a the class of x, we have

$$F = \{b_0 + b_1 a \text{ with } b_0 \text{ and } b_1 \text{ in } \mathbb{F}_7\}.$$

 (a) Determine a formula for $\mathrm{Fr}_7(b_0 + b_1 a)$ of the shape $b_0' + b_1' a$ with b_0' and b_1' in $\mathbb{F}_7$, where Fr_7 is the Frobenius homomorphism in characteristic 7.
 (b) It is given that $E = \mathbb{F}_7[y]/(y^2 + 1)$ is also a field with 49 elements. Find an explicit field isomorphism $\psi : F \to E$, and **briefly explain** why your ψ does the job. *Hint: write elements of E in the form $d_0 + d_1 c$ with c the class of y in E.*

Distribution of points						
1: 8	2a: 5 2b: 9	3: 10	4: 11	5a: 5 5b: 6 5c: 5	6a: 6 6b: 6 6c: 7	7a: 6 7b: 6
Maximum total = 90						
Exam grade = 1 + Total/10						