Faculty of Science	Rings and fields (X_400630), part 1
Vrije Universiteit Amsterdam	Partial examination 27-10-2021 (15:30-17:45)

- **Attempt all problems.**
- Answers without reasoning score poorly, so give proper justifications everywhere.
- In case you cannot do a part of a problem, you may still use its stated result in the remainder of the problem.
- Calculators, notes, books, etc., may not be used.
- Do not hand in scrap, etc., and when handing in $n \geq 1$ sheets, number them $1/n, \ldots, n/n$.

(1) Let k be a field. It is given that $k \times k$ with coordinatewise addition and multiplication is a commutative ring R with identity $1_R \neq 0_R$. Show that every non-zero element of R is either a unit or a zero divisor.

(2) Let $R = \mathbb{Z}[i] = \{a + bi \text{ with } a, b \text{ in } \mathbb{Z}\}$, a subring of $\mathbb{C}$. Use the extended Euclidean algorithm to determine a greatest common divisor d of $\alpha = 6 + 2i$ and $\beta = 3 + 5i$, and to write d in the form $x\alpha + y\beta$ with x and y in R.

(3) **All parts of this problem are independent of each other.**
Let $R = \mathbb{Z}[\sqrt{-3}] = \{a + b\sqrt{-3} \text{ with } a \text{ and } b \text{ in } \mathbb{Z}\}$, a subring of $\mathbb{C}$.

(a) Determine if $\alpha = 3 - \sqrt{-3}$ and $\beta = 3 + 2\sqrt{-3}$ do or do not have a greatest common divisor in R.

(b) For the ideals $I = (2, 1+\sqrt{-3})$ and $J = (2)$ of R, show that $I \neq J$ but $I \cdot I = I \cdot J$.

(c) Show that the ideal $(8, 3 - \sqrt{-3})$ of R is a principal ideal.

(d) Let $D = \{1, 2, 4, 8, \ldots\} = \{2^n \text{ with } n \geq 0\}$, and let S be the ring of fractions $D^{-1}R$. It is given that S has an identity $1_S \neq 0_S$. Determine if $\frac{2+\sqrt{-3}}{2}$ is in S^*.

(4) **In this problem, formulate explicitly the results/theorems/... you use.**
Let $R = \mathbb{Z}[\sqrt{-11}] = \{a + b\sqrt{-11} \text{ with } a \text{ and } b \text{ in } \mathbb{Z}\}$, which is a subring of $\mathbb{C}$, and I the ideal $(9, 4 - \sqrt{-11})$ of R.

(a) Prove that $\varphi : R \to \mathbb{Z}/9\mathbb{Z}$, given by $\varphi(a + b\sqrt{-11}) = \overline{a + 4b}$, is a ring homomorphism with kernel I.

(b) Show that there is a ring isomorphism $R/I \simeq \mathbb{Z}/9\mathbb{Z}$.

(c) Is I a maximal ideal of R? Is it a prime ideal of R?

(5) Let R be the polynomial ring $\mathbb{C}[x]$. In R, we consider its ideals $I = (x^2 + x - 1)$, $J = (x + 1)$ and $K = (x^3 + 2x^2 - 1)$.

(a) Show that there exists a ring isomorphism $R/K \simeq R/I \times R/J$.

(b) Determine $f(x)$ in R with $\deg(f(x)) < 3$ such that $f(x) + K$ is mapped to $(4x + 4 + I, -3 + J)$ under your map in (a).

(6) Let R be a commutative ring with $1 \neq 0$, and a, b elements of R such that the ideal (a, b) of R is equal to R. Prove that $(a^n, b^n) = R$ for each positive integer n.
Hint: $1 = 1^m$ *for any positive integer* m.

Distribution of points					
1: 7	2: 8	3a: 8	4a: 8	5a: 6	6: 8
		3b: 7	4b: 7	5b: 8	
		3c: 8	4c: 7		
		3d: 8			
Maximum total = 90					
Exam grade = 1 + Total/10					