Faculty of Science Vrije Universiteit Amsterdam	Rings and fields (X_400630) Resit 15-2-2021 (15:30-18:15)

- **Attempt all problems.**
- Answers without reasoning score poorly, so give proper justifications everywhere.
- In case you cannot do a part of a problem, you may still use its stated result in the remainder of the problem.
- Calculators, notes, books, etc., may not be used.

(1) Let $R = \mathbb{R} \times \mathbb{R}$ be the product ring, so addition and multiplication are defined coordinatewise. Show that R has an identity $1_R \neq 0_R$, and that every non-zero element of R is a unit or a zero divisor.

(2) Let $R = \mathbb{Z}[i] = \{a + bi \text{ with } a \text{ and } b \text{ in } \mathbb{Z}\}$, a subring of $\mathbb{C}$.
   (a) Use the extended Euclidean algorithm to compute a greatest common divisor of $\alpha = 8 + 6i$ and $\beta = 7 + i$ in R, and to write it in the form $x\alpha + y\beta$ with x and y in R.
   (b) Factorise $5 + 5i$ into irreducibles in R.

(3) Let $R = \mathbb{Z}[\sqrt{-3}] = \{a + b\sqrt{-3} \text{ with } a \text{ and } b \text{ in } \mathbb{Z}\}$, a subring of $\mathbb{C}$. Prove that the ideal $I = (2, 1 + \sqrt{-3})$ is *not* a principal ideal of R. *Hint: use the norm.*

(4) Let $R = \mathbb{Z}[\sqrt{7}] = \{a + b\sqrt{7} \text{ with } a \text{ and } b \text{ in } \mathbb{Z}\}$, which is a subring of $\mathbb{R}$, and I the ideal $(2 - \sqrt{7})$ of R. We define the map $\varphi : R \to \mathbb{Z}/3\mathbb{Z}$ by $\varphi(a + b\sqrt{7}) = \overline{a + 2b}$.
   (a) Show that φ is a ring homomorphism, with kernel I.
   (b) Prove that there is a ring isomorphism $R/I \simeq \mathbb{Z}/3\mathbb{Z}$.
   (c) Determine if $2 - \sqrt{7}$ is, or is not, a prime element of R.

(5) Let R be the polynomial ring $\mathbb{C}[x]$. We define the ideals $I = (x^2 - x + 1)$, $J = (x + 1)$ and $K = (x^3 + 1)$ of R.
   (a) Show that there is a ring isomorphism $R/K \simeq R/I \times R/J$.
   (b) Which element $f(x) + K$ with $\deg(f(x)) < 3$ is mapped to $(x + 1 + I, 6 + J)$ in $R/I \times R/J$?

(6) Let R be a Euclidean domain with norm $M : R \setminus \{0\} \to \{0, 1, 2, 3, \dots\}$, D a subset of $R \setminus \{0\}$ containing 1 that is closed under multiplication, and $S = D^{-1}R$. It is given that S is a domain. We view $R \subseteq S$ by means of the homomorphism $r \mapsto r/1$.
   (a) Show that for $s \neq 0$ in S,

$$N(s) = \min\{M(us) \text{ with } us \text{ in } R \text{ and } u \text{ in } S^*\}$$

   defines an element of $\{0, 1, 2, 3, \dots\}$. *Hint: show some such us exists, and that all such us* $\neq 0$.
   (b) Prove that S with N as norm is a Euclidean domain. *Hint: in order to divide a/d by $s \neq 0$ in S, first divide a by us in R for u in S^* with $N(s) = M(us)$.*

(7) Factorise $3x^4 + 15x + 6$ into irreducibles in $\mathbb{Z}[x]$.

(8) Show that $x^{15}y^{2021} + x^{15}y + y - 1$ is irreducible in $\mathbb{C}[x, y]$. **Formulate the results that you use.**

(9) Let $a = \sqrt{2} + \sqrt{-3}$ and $K = \mathbb{Q}(a)$ in $\mathbb{C}$.
   (a) Prove that $K = \mathbb{Q}(\sqrt{2}, \sqrt{-3})$.
   (b) Determine $[K : \mathbb{Q}]$.

(10) It is given that $f(x) = x^2 + 4x + 2$ is irreducible in $\mathbb{F}_5[x]$, so that $\mathbb{F}_5[x]/(f(x))$ is a field F with 25 elements. With a the class of x, we have

$$F = \{b_0 + b_1 a \text{ with } b_0 \text{ and } b_1 \text{ in } \mathbb{F}_5\}\,.$$

(a) Determine a formula for $\mathrm{Fr}_5(b_0 + b_1 a)$ of the shape $b_0' + b_1' a$ with b_0' and b_1' in $\mathbb{F}_5$, where Fr_5 is the Frobenius homomorphism in characteristic 5.

(b) It is given that $E = \mathbb{F}_5[y]/(y^2 + 2)$ is also a field with 25 elements. Find an explicit field isomorphism $\varphi : F \to E$, and *explain briefly* why your φ does the job. *Hint: write elements of E in the form $d_0 + d_1 c$ with c the class of y in E.*

<table>
<tr><th colspan="10">Distribution of points</th></tr>
<tr><td>1: 3</td><td>2a: 8
2b: 4</td><td>3: 8</td><td>4a: 6
4b: 4
4c: 4</td><td>5a: 5
5b: 6</td><td>6a: 3
6b: 4</td><td>7: 8</td><td>8: 10</td><td>9a: 4
9b: 4</td><td>10a: 4
10b: 5</td></tr>
<tr><th colspan="10">Maximum total = 90</th></tr>
<tr><th colspan="10">Exam grade = 1 + Total/10</th></tr>
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