Faculty of Science	**Rings and fields (X_400630)**
Vrije Universiteit Amsterdam	**Resit 6-2-2020 (18:30-21:15)**

- **Attempt all problems.**
- Answers without reasoning score poorly, so give proper justifications everywhere.
- In case you cannot do a part of a problem, you may still use its stated result in the remainder of the problem.
- Calculators, notes, books, etc., may not be used.

(1) It is given that

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \text{ with } a, b \text{ and } c \text{ in } \mathbb{Z} \text{ and } c \text{ even} \right\}$$

is a subring of $M_2(\mathbb{Z})$, the ring of 2×2 matrices with integral coefficients. Determine the centre

$$Z(R) = \{X \text{ in } R \text{ with } XY = YX \text{ for all } Y \text{ in } R\}$$

of R.

(2) Let $R = \mathbb{Z}[i] = \{a + bi \text{ with } a \text{ and } b \text{ in } \mathbb{Z}\}$, a subring of $\mathbb{C}$.
  (a) Use the extended Euclidean algorithm to compute a greatest common divisor of $\alpha = 2 - 9i$ and $\beta = 4 + 7i$ in R, and to write it in the form $x\alpha + y\beta$ with x and y in R.
  (b) Factorise $3 + 9i$ into irreducibles in R.

(3) Let $R = \mathbb{Z}[\sqrt{-6}] = \{a + b\sqrt{-6} \text{ with } a \text{ and } b \text{ in } \mathbb{Z}\}$, a subring of $\mathbb{C}$.
  (a) Prove that the ideal $I = (3, \sqrt{-6})$ is *not* a principal ideal of R.
  (b) Determine if the elements 3 and $\sqrt{-6}$ have a greatest common divisor in R.
Hint: use the norm.

(4) Let

$$R = \mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \text{ with } a \text{ and } b \text{ in } \mathbb{Z}\},$$

which is a subring of $\mathbb{C}$, and I the ideal $(10, 5 + \sqrt{-5})$ of R. We define the map $\varphi : R \to \mathbb{Z}/10\mathbb{Z}$ by $\varphi(a + b\sqrt{-5}) = \overline{a + 5b}$.
  (a) Show that φ is a ring homomorphism, with kernel I.
  (b) Prove that there is a ring isomorphism $R/I \simeq \mathbb{Z}/10\mathbb{Z}$.
  (c) Determine an ideal J of R such that $I \subsetneq J \subsetneq R$. Do not forget to show that your ideal J satisfies these conditions.

(5) Let R be the polynomial ring $\mathbb{Q}[x]$. We define the ideals $I = (x^2 + x + 2)$, $J = (x - 1)$ and $K = (x^3 + x - 2)$ of R.
  (a) Show that there is a ring isomorphism $R/K \simeq R/I \times R/J$.
  (b) Which element $f(x) + K$ with $\deg(f(x)) < 3$ is mapped to $(-4x - 1 + I, -1 + J)$ in $R/I \times R/J$?

(6) Let

$$R = \left\{ \frac{m}{2^n} \text{ with } m \text{ in } \mathbb{Z} \text{ and } n = 0, 1, 2, 3, \ldots \right\} \subseteq \mathbb{Q}\,.$$

It is given that R is a subring of $\mathbb{Q}$.
  (a) Explain why R is a domain.
  (b) Prove that R is a principal ideal domain. *Hint: if I is an ideal of R show that $J = \{m$ in $\mathbb{Z}$ such that $\frac{m}{1}$ is in $I\}$ is an ideal of $\mathbb{Z}$ and use its generator to generate I.*

(7) Show that each of the following polynomials is irreducible in the given unique factorisation domain. **Formulate the results that you use.**
  (a) $x^3 + 5x + 2$ in $\mathbb{Q}[x]$.
  (b) $x^6 y^{2020} - xy - y + x + 1$ in $\mathbb{R}[x, y]$.

(8) Let $\zeta = e^{2\pi i/3}$, $a = \sqrt[3]{2}$, $K = \mathbb{Q}(a)$ and $F = K(\zeta)$ in $\mathbb{C}$.
  (a) Prove that $[K : \mathbb{Q}] = 3$.
  (b) Now show that $[F : \mathbb{Q}] = 6$.
  Hint: consider minimal polynomials.

(9) It is given that $f(x) = x^2 + 3x + 3$ is irreducible in $\mathbb{F}_5[x]$, so that $\mathbb{F}_5[x]/(f(x))$ is a field F with 25 elements. With a the class of x, we have

$$F = \{b_0 + b_1 a \text{ with } b_0 \text{ and } b_1 \text{ in } \mathbb{F}_5\}.$$

  (a) Determine a formula for $\mathrm{Fr}_5(b_0 + b_1 a)$ of the shape $b_0' + b_1' a$ with b_0' and b_1' in $\mathbb{F}_5$, where Fr_5 is the Frobenius homomorphism in characteristic 5.
  (b) It is given that $E = \mathbb{F}_5[y]/(y^2 + 3)$ is also a field with 25 elements. Find an explicit field isomorphism $\varphi : F \to E$, and *explain briefly* why your φ does the job. *Hint: write elements of E in the form $d_0 + d_1 c$ with c the class of y in E.*

Distribution of points								
1: 4	2a: 5	3a: 5	4a: 6	5a: 5	6a: 2	7a: 8	8a: 4	9a: 4
	2b: 5	3b: 4	4b: 4	5b: 5	6b: 6	7b: 10	8b: 3	9b: 5
			4c: 5					
Maximum total = 90								
Exam grade = 1 + Total/10								