Short Test 1 – Probability Theory course 2024

Friday February 23rd, 9:45–10:30 (45 minutes)

Full name: **Student number:** **TA name:**

- Write your full name and student number on both sheets provided, together with the name of your TA.
- **This pair of sheets is the only one that you need to hand in to your TA at the end of the test.** Please write your answers *starting with the back of this sheet.* If you run out of space, ask the TA for an extra sheet and do not forget to also write there your name and student number. Scrap paper is provided upon request. Any solution written on scrap paper will not be graded; make sure to copy it on these sheets.
- **No material of any kind is allowed during the short test.** In particular, no book and no notes.
- **Smartphones are not allowed.** You may use a **simple calculator** for this test, but this is not necessary. In fact, you do not have to work out powers, fractions, binomial coefficients, products, etc.
- Use proper notation and terminology, justify in a clear way all your steps, and do not just write the final numerical answer.
- For any theorem, property, or formula you use, before applying it clearly state its name (if any) and its general expression.
- You have 45 minutes available for the test. You cannot leave the short test before 10:15. If you want to leave before the end of the test, raise your hand, and wait for the TA to collect your sheets.
- Your short test grade is given by 1+ the number of points (thus 10 is the maximum), rounded to the 1st decimal.

Exercise 1 [2.5 points]

Two six-faced fair dice are rolled simultaneously.

(a) [0.6 points] Describe in detail the corresponding probability space.

(b) [0.4 points] What is the probability that at least one is a six?

(c) [1 point] If the two faces are different, what is the probability that at least one is a six?

(d) [0.5 points] If at least one is a six, what is the probability that the two faces are different?

Solution

(a) $\Omega = \{(a,b) \ : \ a,b \in \{1,\dots,6\}\}$ consists of all ordered pairs with numbers between 1 and 6, $\mathcal{F} = 2^\Omega$ is the collection of all subsets of Ω, and all outcomes are equally likely, i.e., $\mathbb{P}(\omega) = 1/\#\Omega \ \forall \omega \in \Omega$.

(b) Let $E = \{\text{at least one 6}\} = \{(1,6),(2,6),(3,6),(4,6),(5,6),(6,6),(6,1),(6,2),(6,3),(6,4),(6,5)\} \subseteq \Omega$, which is a subset containing 11 elements. Then $\mathbb{P}(E) = \#E/\#\Omega = 11/36$.

(c) Let $D = \{\text{two faces are different}\}$, which is a subset of Ω consisting of $30 = 6 \cdot 5$ elements. Then $\mathbb{P}(D) = \#D/\#\Omega = 30/36 = 5/6$.

The event $E \cap D = \{\text{two faces are different and at least one is a six}\}$ consists of all elements of E except the pair $(6,6)$, so it has 10 elements in total, which implies that $\mathbb{P}(E \cap D) = \#(E \cap D)/\#\Omega = 10/36$.

Therefore, using the definition of conditional probability, we get

$$\mathbb{P}(E|D) = \frac{\mathbb{P}(E \cap D)}{\mathbb{P}(D)} = \frac{10/36}{5/6} = \frac{1}{3}.$$

(d) Using again the definition of conditional probability and the previously calculated quantities, we get

$$\mathbb{P}(D|E) = \frac{\mathbb{P}(E \cap D)}{\mathbb{P}(E)} = \frac{10/36}{11/36} = \frac{10}{11}.$$

Exercise 2 [3.25 points]

We draw the top 7 cards from a well-shuffled standard 52-card deck. Find the probability that:

(a) [1 point] The 7 cards include exactly 3 aces.

(b) [1 point] The 7 cards include at least 3 queens.

(c) [1.25 point] Calculate the probability that the 7th card is the first king to be dealt.

Solution

(a) The sample space consists of all ways of drawing 7 elements out of a set of 52 (without replacement and the order does not matter), so there are $\binom{52}{7}$ possible outcomes, all equally likely.

Let us count those outcomes that involve exactly 3 aces. We are free to select any 3 out of the 4 aces, and any 4 out of the 48 remaining cards, for a total of $\binom{4}{3}\binom{48}{4}$ choices. Thus,

$$\mathbb{P}(\text{7 cards include exactly 3 aces}) = \frac{\binom{4}{3}\binom{48}{4}}{\binom{52}{7}} = \frac{9}{1547} \approx 0.00582.$$

(b) We decompose the event into two disjoint events

$$\{\text{7 cards include exactly 3 queens}\} \cup \{\text{7 cards include exactly 4 queens}\}.$$

Using this fact and the additivity property of $\mathbb{P}$, we get

$$\begin{aligned}\mathbb{P}(\text{7 cards include at least 3 queens}) &= \mathbb{P}(\text{7 cards include 3 queens}) + \mathbb{P}(\text{7 cards include 4 queens}) \\ &= \frac{\binom{4}{3}\binom{48}{4}}{\binom{52}{7}} + \frac{\binom{4}{4}\binom{48}{3}}{\binom{52}{7}} = \frac{46}{7735} \approx 0.00595.\end{aligned}$$

(c) Let A_i the event that the i-th card is **not** a king. We need to compute the probability of the event $A_1 \cap A_2 \cap \ldots \cap A_6 \cap A_7^c$.

Using the generalized multiplication rule (fact 2.6) , we get

$$\begin{aligned}\mathbb{P}(A_1 \cap A_2 \cap \ldots \cap A_6 \cap A_7^c) &= \mathbb{P}(A_1) \cdot \mathbb{P}(A_2|A_1) \cdot \mathbb{P}(A_3|A_1 \cap A_2) \cdot \ldots \cdot \mathbb{P}(A_7^c|A_1 \cap A_2 \cap \ldots \cap A_6) \\ &= \frac{48}{52} \cdot \frac{47}{51} \cdot \frac{46}{50} \cdot \frac{45}{49} \cdot \frac{44}{48} \cdot \frac{43}{47} \cdot \frac{4}{46} = \frac{2838}{54145} \approx 0.0524.\end{aligned}$$

The second step can be justified as follows. The probability that the first card is not a king is 48/52. Given that, the probability that the second is not a king is 47/51 (there are 47 non-king cards left among the 51 remaining). We continue similarly until the 6-th card. The probability that the 6-th card is not a king, given that none of the preceding five was a king, is 43/47 (there are $52 - 5 = 47$ cards remaining, and $48 - 5 = 43$ of them are not kings). Finally, the conditional probability that the 7-th card is a king is 4/46 (there are 4 kings among the 46 remaining cards).

Exercise 3 [0.75 points]

Consider two events A, B such that $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$. If the occurrence of B makes A more likely, does the occurrence of A make B more likely? In other words, if $\mathbb{P}(A|B) > \mathbb{P}(A)$, is it true that $\mathbb{P}(B|A) > \mathbb{P}(B)$? If yes, prove it; if not, provide a simple counterexample.

Solution

Yes. By virtue of the definition of conditional probability, $\mathbb{P}(A|B) > \mathbb{P}(A)$ is equivalent to $\mathbb{P}(A \cap B) > \mathbb{P}(A)\mathbb{P}(B)$, which is equivalent to $\mathbb{P}(B|A) > \mathbb{P}(B)$.

Exercise 4 [2.5 points]

You enter a chess tournament where your probability of winning a game is 0.3 against the high-rank players (50% of the total), 0.4 against the mid-rank players (25% of the total), and 0.5 against the low-rank players (25% of the total). You play a game against a randomly chosen opponent.

(a) [1 point] What is the probability of winning?

(b) [1.5 points] Suppose that you win the game. What is the probability that you played against a high-rank player?

Solution

(a) Let O_h, O_m, O_l the events that you play against a high-rank, mid-rank and low-rank opponent, respectively. The description of the problem implies that

$$\mathbb{P}(O_h) = 0.5, \quad \mathbb{P}(O_m) = 0.25, \quad \text{and} \quad \mathbb{P}(O_m) = 0.25.$$

Let W be the event of winning. From the problem description, it follows that

$$\mathbb{P}(W|O_h) = 0.3, \quad \mathbb{P}(W|O_m) = 0.4, \quad \text{and} \quad \mathbb{P}(W|O_l) = 0.5.$$

By the law of total probability (fact 2.10), the probability of winning is

$$\mathbb{P}(W) = \mathbb{P}(W|O_h)\mathbb{P}(O_h) + \mathbb{P}(W|O_m)\mathbb{P}(O_m) + \mathbb{P}(W|O_l)\mathbb{P}(O_l) = 0.5 \cdot 0.3 + 0.25 \cdot 0.4 + 0.25 \cdot 0.5 = 0.375.$$

(b) The three events O_h, O_m, O_l all have strictly positive probability, as outlined in (a), and partition the sample space. Since $\mathbb{P}(W) > 0$, using the general version of Bayes' theorem (fact 2.15), we have

$$\begin{aligned}
\mathbb{P}(O_h|W) &= \frac{\mathbb{P}(W|O_h)\mathbb{P}(O_h)}{\mathbb{P}(W|O_h)\mathbb{P}(O_h) + \mathbb{P}(W|O_m)\mathbb{P}(O_m) + \mathbb{P}(W|O_l)\mathbb{P}(O_l)} \\
&= \frac{0.5 \cdot 0.3}{0.5 \cdot 0.3 + 0.25 \cdot 0.4 + 0.25 \cdot 0.5} = 0.4.
\end{aligned}$$