Probability Theory – Midterm Exam 2023

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27th March 2023, 12:15 – 14:15 (+30 min for extra timers)

The exam consists of six exercises worth a total of 95 *points* $= 16 + 11 + 7 + 25 + 21 + 15$. *The midterm exam grade is calculated as* `1 + (your points)/10` *(the maximum is thus* 10.5*). Note that the last three exercises are worth more points than the first three. The exam lasts for* $2h$ *(+* 30 *min for extra timers). Do not forget to add the name of the formula you are using and/or explanations next to your algebraic/numerical solutions. No book and no notes, you are allowed to use only a* simple *calculator.*

Exercise 1 [16 points = 7 + 4 + 5]

From a shuffled poker deck of 52 cards (13 for each suit), we draw three cards at random.
Note*: in this problem, you do not have to simplify powers/fractions/binomials nor approximate them.*

(a) [7 points] Describe the sample space Ω first and then calculate the probability that at least one of the three cards is an ace.

(b) [4 points] Calculate the probability that the three cards drawn are of three different suits.

(c) [5 points] Calculate the probability that at least two of the cards drawn have the same number or the same face.

Exercise 2 [11 points = 6 + 5]

An insurance company offers a policy that provides for the payment of a lump sum for damage suffered by the customer. The company classifies policyholders into three categories: "low risk", "medium risk" and "high risk". Of its policyholders, 75% are "low risk", 20% are "medium risk" and the remaining 5% are "high risk". It is known that "low risk" policyholders have a 2% probability of suffering damage that requires payment of the insurance, while this probability is 10% for "medium risk" policyholders and 20% for those at "high risk".

(a) [6 points] Rephrase the provided information using appropriate events and conditional probabilities and use them to calculate the probability that a randomly selected individual among the policyholders requires insurance payment.

(b) [5 points] If a policyholder requires insurance payment, what is the probability that she/he is in the "high risk" category?

Exercise 3 [7 points = 3 + 4]

An airline knows that 5% of the people making reservations on a certain flight will not show up. Consequently, their policy is to sell 52 tickets for a flight that can hold only 50 passengers. Assume that for a given flight all 52 tickets were sold and let X be the random variable describing the number of passengers that show up for it.

(a) [3 points] What is the name of the distribution of X? What are its parameters? Calculate $\mathbb{E}(X)$.

(b) [4 points] What is the probability that there will be a seat available for every passenger who shows up? Note*: in this question you do not have to simplify powers/fractions/binomials nor approximate them.*

Do not forget to turn the page!

Exercise 4 [25 points = 4 + 4 + 5 + 5 + 7]

An urn contains 12 marbles of which 3 are red, 3 yellow, 3 green, and 3 blue. We select 3 arbitrary marbles from this urn without replacement (and without order, so we take the 3 marbles at the same moment). Let A be the event that at least one of the marbles is yellow and B be the event that at least one of the marbles is red.

(a) [4 points] Show that $\mathbb{P}(A) = \mathbb{P}(B) = \frac{34}{55}$.

(b) [4 points] Show that $\mathbb{P}(A \cup B) = \frac{10}{11}$ by first computing the probability of the complement of $A \cup B$.

(c) [5 points] Compute $\mathbb{P}(A \cap B)$ using the answers of (a) and (b). Are A and B independent?

We put all marbles back in the urn. We now select arbitrary marbles one by one from the urn and without replacement, until we have obtained 2 red marbles in total. Let X be the number of marbles we take and let C be the event that the second marble selected is red.

(d) [5 points] Compute $\mathbb{P}(X = 5)$.

(e) [7 points] Compute $\mathbb{P}(X = 3 \mid C)$.

Exercise 5 [21 points = 3 + 7 + 7 + 4]

Let X be a continuous random variable with density function

$$f(x) = \begin{cases} \frac{2}{x^2} & \text{if } 1 \leq x \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

(a) [3 points] Check that f is a probability density function.

(b) [7 points] Compute $\mathbb{P}\left(1 < X < \frac{5}{4} \mid X < \frac{3}{2}\right)$.

(c) [7 points] Compute $\mathrm{Var}(X)$.

(d) [4 points] Let $Y := \frac{1}{X^2}$. Compute $\mathbb{E}(Y)$.

Exercise 6 [15 points = 8 + 7]

For this question, you need the table provided on the next page. The diameter of a 1 euro coin is normally distributed with an expectation of 23.25 mm and a standard deviation of 0.10 mm. A vending machine accepts only 1 euro coins with a diameter between 22.95 mm and 23.45 mm.

(a) [8 points] Compute the probability that the vending machine does not accept a euro coin.

(b) [7 points] Let X be the diameter (in mm) of a 1 euro coin. Compute for which value d we have $\mathbb{P}(|X - 23.25| < d) \approx 0.95$.

Appendix E

Table of values for $\Phi(x)$

$\Phi(x) = P(Z \leq x)$ is the cumulative distribution function of the standard normal random variable Z.

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998