Probability Theory – Midterm Exam 2023 (Solutions)

dr. A. Zocca

27th March 2023, 12:15 – 14:15 (+30 min for extra timers)

The exam consists of six exercises worth a total of 95 points = 16 + 11 + 7 + 25 + 21 + 15. The midterm exam grade is calculated as `1 + (your points)/10` *(the maximum is thus 10.5). Note that the last three exercises are worth more points than the first three. The exam lasts for 2h (+ 30 min for extra timers). Do not forget to add the name of the formula you are using and/or explanations next to your algebraic/numerical solutions. No book and no notes, you are allowed to use only a simple calculator.*

Exercise 1 [16 points = 7 + 4 + 5]

From a shuffled poker deck of 52 cards (13 for each suit), we draw three cards at random.
Note: in this problem, you do not have to simplify powers/fractions/binomials nor approximate them.

(a) [7 points] Describe the sample space Ω first and then calculate the probability that at least one of the three cards is an ace.

The sample space Ω is the collection of unordered subsets of three cards (hence without repetitions), therefore $|\Omega| = \binom{52}{3}$.

Let A be the event we need to compute the probability. Its complement A^c is the event in which none of the three picked cards is an ace. The number of card triplets in A^c is $\binom{48}{3}$, therefore the required probability is then

$$\mathbb{P}(A) = 1 - \mathbb{P}(A^c) = 1 - \frac{\binom{48}{3}}{\binom{52}{3}} \left(= \frac{1201}{5525} \approx 0.217. \right)$$

(b) [4 points] Calculate the probability that the three cards drawn are of three different suits.

Let B be the event we need to compute the probability. Picking an element from B means choosing 3 out of 4 suits (in $\binom{4}{3}$ ways) and then picking each of the 3 cards, each of which can be independently picked in 13 different ways. Therefore $|B| = \binom{4}{3} \cdot 13^3$, and the required probability is then

$$\mathbb{P}(B) = \frac{|B|}{|\Omega|} = \frac{\binom{4}{3} \cdot 13^3}{\binom{52}{3}} \left(= \frac{169}{425} \approx 0.398. \right)$$

(c) [5 points] Calculate the probability that at least two of the cards drawn have the same number or the same face.

Let C be the event we need to compute the probability. Then, its complement C^c is the event in which the three cards drawn are all different numbers/faces. Picking an element from C^c means picking a number/face among the 13 available (there are $\binom{13}{3}$ ways of doing so). Once the three numbers/faces are fixed, we can pick each card in any of the 4 suits, so $|C^c| = \binom{13}{3} \cdot 4^3$, and the required probability is then

$$\mathbb{P}(C) = 1 - \mathbb{P}(C^c) = 1 - \frac{|C^c|}{|\Omega|} = 1 - \frac{\binom{13}{3} \cdot 4^3}{\binom{52}{3}} \left(= \frac{73}{425} \approx 0.172. \right)$$

Exercise 2 [11 points = 6 + 5]

An insurance company offers a policy that provides for the payment of a lump sum for damage suffered by the customer. The company classifies policyholders into three categories: "low risk", "medium risk" and "high risk". Of its policyholders, 75% are "low risk", 20% are "medium risk" and the remaining 5% are "high risk". It is known that "low risk" policyholders have a 2% probability of suffering damage that requires payment of the insurance, while this probability is 10% for "medium risk" policyholders and 20% for those at "high risk".

(a) [6 points] Rephrase the provided information using appropriate events and conditional probabilities and use them to calculate the probability that a randomly selected individual among the policyholders requires insurance payment.

Consider the events:

- $A_1 = \{$ the chosen policyholder is "low risk" $\}$,
- $A_2 = \{$ the chosen policyholder is "medium risk" $\}$,
- $A_3 = \{$ the chosen policyholder is "high risk" $\}$,
- $B = \{$ a policyholder claims an insurance payment $\}$,

From the problem description, we know that

$\mathbb{P}(B \mid A_1) = 0.02$, $\mathbb{P}(B \mid A_2) = 0.1$, $\mathbb{P}(B \mid A_3) = 0.2$, $\mathbb{P}(A_1) = 0.75$, $\mathbb{P}(A_2) = 0.2$, $\mathbb{P}(A_3) = 0.05$.

We can calculate

$$\mathbb{P}(B) = \sum_{i=1}^{3} \mathbb{P}(B \mid A_i)\,\mathbb{P}(A_i) = 0.02 \cdot 0.75 + 0.1 \cdot 0.2 + 0.2 \cdot 0.05 = 0.045.$$

(b) [5 points] If a policyholder requires insurance payment, what is the probability that she/he is in the "high risk" category?

Using Bayes' formula, we get

$$\mathbb{P}(A_3 \mid B) = \frac{\mathbb{P}(B \mid A_3)\,\mathbb{P}(A_3)}{\mathbb{P}(B)} = \frac{0.2 \cdot 0.05}{0.045} = \frac{2}{9} (\approx 0.222).$$

Exercise 3 [7 points = 3 + 4]

An airline knows that 5% of the people making reservations on a certain flight will not show up. Consequently, their policy is to sell 52 tickets for a flight that can hold only 50 passengers. Assume that for a given flight all 52 tickets were sold and let X be the random variable describing the number of passengers that show up for it.

(a) [3 points] What is the name of the distribution of X? What are its parameters? Calculate $\mathbb{E}(X)$.

X is a binomial distribution $\text{Bin}(n, p)$ with $n = 52$ and $p = 0.95$. $\mathbb{E}(X) = n \cdot p = 52 \cdot 0.95 = 49.4$.

(b) [4 points] What is the probability that there will be a seat available for every passenger who shows up? <u>*Note*</u>*: in this question you do not have to simplify powers/fractions/binomials nor approximate them.*

The required probability is $\mathbb{P}(X \leq 50)$. It can be calculated as follows

$$\mathbb{P}(X \leq 50) = 1 - \mathbb{P}(X = 51) - \mathbb{P}(X = 52) = 1 - \binom{52}{52}(0.95)^{52}(0.05)^{0} - \binom{52}{51}(0.95)^{51}(0.05) \, (\approx 0.740).$$

Exercise 4 [25 points = 4 + 4 + 5 + 5 + 7]

An urn contains 12 marbles of which 3 are red, 3 yellow, 3 green, and 3 blue. We select 3 arbitrary marbles from this urn without replacement (and without order, so we take the 3 marbles at the same moment). Let A be the event that at least one of the marbles is yellow and B be the event that at least one of the marbles is red.

(a) [4 points] Show that $\mathbb{P}(A) = \mathbb{P}(B) = \frac{34}{55}$.

$$\mathbb{P}(A) = 1 - \mathbb{P}(A^c) = 1 - \mathbb{P}(\text{no yellow marbles}) = 1 - \frac{\binom{9}{3}}{\binom{12}{3}} = 1 - \frac{84}{220} = \frac{34}{55}.$$

For $\mathbb{P}(B)$ one can argue exactly in the same way.

(b) [4 points] Show that $\mathbb{P}(A \cup B) = \frac{10}{11}$ by first computing the probability of the complement of $A \cup B$.

$(A \cup B)^c = A^c \cap B^c = \{\text{no red marbles and no yellow marbles}\} = \{\text{only green and blue marbles}\}$

$$\mathbb{P}(A \cup B) = 1 - \mathbb{P}((A \cup B)^c) = 1 - \frac{\binom{6}{3}}{\binom{12}{3}} = 1 - \frac{20}{220} = \frac{10}{11}.$$

(c) [5 points] Compute $\mathbb{P}(A \cap B)$ using the answers of (a) and (b). Are A and B independent?

We can use the following formula $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.
After we re-arrange it to obtain $\mathbb{P}(A \cap B)$, we can calculate

$$\begin{aligned}\mathbb{P}(A \cap B) &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B)\\ &= \frac{34}{55} + \frac{34}{55} - \frac{10}{11} = \frac{18}{55}.\end{aligned}$$

Since $\mathbb{P}(A \cup B) = \frac{18}{55} \neq \frac{34}{55} \cdot \frac{34}{55} = \mathbb{P}(A) \cdot \mathbb{P}(B)$, the two events are not independent.

We put all marbles back in the urn. We now select arbitrary marbles one by one from the urn and without replacement, until we have obtained 2 red marbles in total. Let X be the number of marbles we take and let C be the event that the second marble selected is red.

(d) [5 points] Compute $\mathbb{P}(X = 5)$.

$$\mathbb{P}(X = 5)$$

$$= \mathbb{P}(\text{one red in the first four and 5th marble red})$$

$$= \mathbb{P}(\text{one red in the first four})\, \mathbb{P}(\text{5th marble red} \mid \text{one red in the first four})$$

$$= \frac{\binom{3}{1}\binom{9}{3}}{\binom{12}{4}} \cdot \frac{2}{8} = \frac{3 \cdot 84}{495} \cdot \frac{1}{4} = \frac{7}{55}$$

(e) [7 points] Compute $\mathbb{P}(X = 3 \mid C)$.

$$\mathbb{P}(C) =$$

$$= \mathbb{P}(\text{1st red, 2nd red}) + \mathbb{P}(\text{1st not red, 2nd red})$$

$$= \frac{1}{4} \cdot \frac{2}{11} + \frac{3}{4} \cdot \frac{3}{11} = \frac{11}{44} = \frac{1}{4}.$$

Furthermore,

$$\mathbb{P}(\{X = 3\} \cap C) =$$

$$= \mathbb{P}(\text{1st not red, 2nd red, 3rd red})$$

$$= \frac{3}{4} \cdot \frac{3}{11} \cdot \frac{2}{10} = \frac{9}{220}.$$

Lastly, $\mathbb{P}(X = 3 \mid C) = \dfrac{\mathbb{P}(\{X = 3\} \cap C)}{\mathbb{P}(C)} = \dfrac{18}{110} = \dfrac{9}{55}$.

Exercise 5 [21 points = 3 + 7 + 7 + 4]

Let X be a continuous random variable with density function

$$f(x) = \begin{cases} \frac{2}{x^2} & \text{if } 1 \leq x \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

(a) [3 points] Check that f is a probability density function.

$f(x) \geq 0$ for all $x \in \mathbb{R}$ (by definition and by nonnegativity of the square at the denominator) and

$$\int_{-\infty}^{+\infty} f(x)\, dx = \int_1^2 \frac{2}{x^2}\, dx = \left[-\frac{2}{x}\right]_1^2 = -1 - (-2) = 1.$$

(b) [7 points] Compute $\mathbb{P}\left(1 < X < \frac{5}{4} \mid X < \frac{3}{2}\right)$.

$$\mathbb{P}\left(1 < X < \frac{5}{4} \mid X < \frac{3}{2}\right) = \frac{\mathbb{P}\left(\{1 < X < \frac{5}{4}\} \cap \{X < \frac{3}{2}\}\right)}{\mathbb{P}\left(X < \frac{3}{2}\right)} = \frac{\mathbb{P}\left(\{1 < X < \frac{5}{4}\}\right)}{\mathbb{P}\left(X < \frac{3}{2}\right)}.$$

$$\mathbb{P}\left(1 < X < \frac{5}{4}\right) = \int_1^{\frac{5}{4}} \frac{2}{x^2}\, dx = \left[-\frac{2}{x}\right]_1^{\frac{5}{4}} = -\frac{8}{5} - (-2) = \frac{2}{5}.$$

$$\mathbb{P}\left(X < \frac{3}{2}\right) = \int_1^{\frac{3}{2}} \frac{2}{x^2}\, dx = \left[-\frac{2}{x}\right]_1^{\frac{3}{2}} = -\frac{4}{3} - (-2) = \frac{2}{3}.$$

Therefore $\mathbb{P}\left(1 < X < \frac{5}{4} \mid X < \frac{3}{2}\right) = \frac{\frac{2}{5}}{\frac{2}{3}} = \frac{3}{5}$.

(c) [7 points] Compute $\mathrm{Var}(X)$.

We can compute the first two moments of the random variable X as follows:

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} x \cdot f(x)\, dx = \int_1^2 x \cdot \frac{2}{x^2}\, dx = \int_1^2 \frac{2}{x}\, dx = [2\ln(x)]_1^2 = 2\ln(2)$$

$$\mathbb{E}\left(X^2\right) = \int_{-\infty}^{+\infty} x^2 \cdot f(x)\,dx = \int_1^2 2\,dx = 2.$$

Therefore, $\mathrm{Var}\left(X\right) = \mathbb{E}\left(X^2\right) - (\mathbb{E}\left(X\right))^2 = 2 - 4(\ln(2))^2$.

(d) [4 points] Let $Y := \frac{1}{X^2}$. Compute $\mathbb{E}\left(Y\right)$.

Let $g(x) := \frac{1}{x^2}$. Then, using the formula for $\mathbb{E}\left(g(X)\right)$, we can calculate

$$\mathbb{E}\left(Y\right) = \mathbb{E}\left(g(X)\right) = \int_{-\infty}^{+\infty} g(x) \cdot f(x)\,dx = \int_1^2 \frac{2}{x^4}\,dx = \left[-\frac{2}{3 \cdot x^3}\right]_1^2 = -\frac{2}{24} + \frac{2}{3} = \frac{7}{12}.$$

Exercise 6 [15 points = 8 + 7]

For this question, you need the table provided on the next page. The diameter of a 1 euro coin is normally distributed with an expectation of 23.25 mm and a standard deviation of 0.10 mm. A vending machine accepts only 1 euro coins with a diameter between 22.95 mm and 23.45 mm.

(a) [8 points] Compute the probability that the vending machine does not accept a euro coin.

We need to calculate $\mathbb{P}\left(X < 22.95 \text{ or } X > 23.45\right)$

$$\mathbb{P}\left(X < 22.95 \text{ or } X > 23.45\right) =$$

$$= \mathbb{P}\left(X < 22.95\right) + \mathbb{P}\left(X > 23.45\right)$$

$$= \mathbb{P}\left(\frac{X - 23.25}{0.1} < \frac{22.95 - 23.25}{0.1}\right) + \mathbb{P}\left(\frac{X - 23.25}{0.1} > \frac{23.45 - 23.25}{0.1}\right)$$

$$= \mathbb{P}\left(Z < -3\right) + \mathbb{P}\left(Z > 2\right)$$

$$= (1 - \Phi(3)) + (1 - \Phi(2))$$

$$= 2 - 0.9987 - 0.9772 = 0.0241.$$

(b) [7 points] Let X be the diameter (in mm) of a 1 euro coin. Compute for which value d we have $\mathbb{P}\left(|X - 23.25| < d\right) \approx 0.95$.

$$\mathbb{P}\left(|X - 23.25| < d\right) = \mathbb{P}\left(-d < X - 23.25 < d\right) = \mathbb{P}\left(\frac{-d}{0.1} < \frac{X - 23.25}{0.1} < \frac{d}{0.1}\right)$$

$$= \Phi\left(\frac{d}{0.1}\right) - \Phi\left(\frac{-d}{0.1}\right) = \Phi\left(\frac{d}{0.1}\right) - \left(1 - \Phi\left(\frac{d}{0.1}\right)\right) = 2\Phi\left(\frac{d}{0.1}\right) - 1.$$

Therefore, $2\Phi\left(\frac{d}{0.1}\right) - 1 \approx 0.95$ if and only if $\Phi\left(\frac{d}{0.1}\right) \approx 0.975$. By looking at the table we get $\frac{d}{0.1} \approx 1.96$, and thus $d \approx 0.196$.