Short Test 2 – Probability Theory course 2023 – Solutions

Time available: 30 minutes

The short test grade is given by 1+ *the number of points (thus* 10 *is the maximum), rounded to the 1st decimal.*

Exercise 1 [3 points]

Consider a continuous random variable X with probability density function

$$f_X(x) := \begin{cases} c \cdot (4x - 2x^2) & \text{if } 0 < x < 2, \\ 0 & \text{otherwise.} \end{cases}$$

(a) [1 point] Find the correct value of c.

(b) [1 point] Give the corresponding cumulative distribution function F_X.

(c) [0.5 point] Calculate $\mathbb{P}\left(\frac{1}{2} < X < \frac{3}{2}\right)$.

(d) [0.5 point] Calculate $\mathbb{P}\left(X = \frac{2}{3}\right)$ and $\mathbb{P}\left(X = \frac{5}{6}\right)$.

Solution

(a) The density f_X is always non-negative if $c \geq 0$, since $(4x - 2x^2) = 2x(2 - x)$ is always nonnegative in the interval $0 < x < 2$. However,

$$\int_{-\infty}^{+\infty} f_X(x)\, dx = c \int_0^2 (4x - 2x^2)\, dx = c \cdot \left[2x^2 - \frac{2}{3}x^3\right]_0^2 = \frac{8c}{3},$$

so c must be equal to $\frac{3}{8}$ for f_X to be a density.

(b) The cumulative distribution function can be calculated as

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt.$$

If $x < 0$, then clearly $F_X(x) = 0$. If $0 \leq x \leq 2$, then

$$F_X(x) = \int_{-\infty}^{x} f(t)\, dt = \int_0^x c \cdot (4t - 2t^2)\, dt = c \cdot \left[2t^2 - \frac{2}{3}t^3\right]_0^x = c \cdot (2x^2 - \frac{2}{3}x^3) = \frac{3}{4}x^2 - \frac{1}{4}x^3.$$

Lastly, if $x > 2$, then $F_X(x) = 1$.

(c) $\mathbb{P}\left(\frac{1}{2} < X < \frac{3}{2}\right) = \int_{\frac{1}{2}}^{\frac{3}{2}} c \cdot (4x - 2x^2)\, dx = F_X\left(\frac{3}{2}\right) - F_X\left(\frac{1}{2}\right) = \frac{27}{32} - \frac{5}{32} = \frac{11}{16}.$

(d) Both probabilities are equal to zero since X is a continuous random variable, which implies that the probability of every singleton is equal to 0.

Exercise 2 [6 points]

Jasmine has three children, each of which is equally likely to be a boy or a girl, independently of the others. Consider the events:

$$A = \{\text{all the children are of the same sex}\}$$
$$B = \{\text{there is at most one boy}\}$$
$$C = \{\text{the family includes at least a boy and a girl}\}$$

(a) [1.75 points] Show that A is independent of B and that B is independent of C.

(b) [0.5 point] Is A independent of C? Justify your answer extensively in either case.

Assuming now that Jasmine has still three children, but the probability of each child being a girl is equal to $\frac{3}{5}$, independently of the others. Let X be the number of Jasmine's daughters.

(c) [0.75 points] What is the distribution of X? Give the probability mass function of X.

(d) [1 point] Consider the events B and C as defined above and express them using the random variable X. Is B independent of C? Justify your answer extensively in either case.

Jasmine has 11 male cousins and 9 female cousins. During a family gathering where there all present, 6 out of these 20 cousins are chosen at random to form a volleyball team. Denote by Y the number of female players in such a team.

(e) [1 point] What is the distribution of Y? Specify the probability mass function of Y.

(f) [1 point] What is the probability that the team consists of at least 5 female players?

Solution

We use the notation G for a girl and B for a boy. In the following, all events of the form $xG, (3-x)B$ should be considered unordered sets of children.

(a) $\mathbb{P}(A) = \mathbb{P}(3G) + \mathbb{P}(3B) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$
$\mathbb{P}(B) = \mathbb{P}(3G) + \mathbb{P}(1B, 2G) = \frac{1}{8} + \binom{3}{1}\frac{1}{8} = \frac{1}{2}$.
$A \cap B$ is the event $\{3G\}$, which has probability $(\frac{1}{2})^3 = \frac{1}{8}$.
A and B are independent since $\mathbb{P}(A \cap B) = \frac{1}{8} = \frac{1}{4} \cdot \frac{1}{2} = \mathbb{P}(A) \cdot \mathbb{P}(B)$.

Since $C = A^c$, we have $\mathbb{P}(C) = 1 - \mathbb{P}(A) = \frac{3}{4}$
$B \cap C$ is the event $\{1B, 2G\}$, which has probability $\binom{3}{1}(\frac{1}{2})^3 = \frac{3}{8}$.
B and C are also independent since $\mathbb{P}(B \cap C) = \frac{3}{8} = \frac{1}{2} \cdot \frac{3}{4} = \mathbb{P}(B) \cdot \mathbb{P}(C)$.

(b) A and C are not independent since

$$\mathbb{P}(A \cap C) = \mathbb{P}(\emptyset) = 0 \neq \frac{1}{4} \cdot \frac{3}{4} = \mathbb{P}(A) \cdot \mathbb{P}(C).$$

(c) X is a binomial distribution with parameters $n = 3$ and $p = \frac{3}{5}$. The probability mass function of X is:

$$\mathbb{P}(X=0) = \binom{3}{0}\left(\frac{2}{5}\right)^3 = \frac{8}{125} \qquad \mathbb{P}(X=1) = \binom{3}{1}\left(\frac{3}{5}\right)^1\left(\frac{2}{5}\right)^2 = \frac{36}{125}$$
$$\mathbb{P}(X=2) = \binom{3}{2}\left(\frac{3}{5}\right)^2\left(\frac{2}{5}\right) = \frac{54}{125} \qquad \mathbb{P}(X=3) = \binom{3}{3}\left(\frac{3}{5}\right)^3 = \frac{27}{125}.$$

(d) $\mathbb{P}(B) = \mathbb{P}(X = 3) + \mathbb{P}(X = 2) = \binom{3}{3}\left(\frac{3}{5}\right)^3 + \binom{3}{2}\left(\frac{3}{5}\right)^2\left(\frac{2}{5}\right) = \frac{27}{125} + \frac{54}{125} = \frac{81}{125}$.
$\mathbb{P}(C) = \mathbb{P}(X = 1) + \mathbb{P}(X = 2) = \binom{3}{1}\left(\frac{3}{5}\right)\left(\frac{2}{5}\right)^2 + \binom{3}{2}\left(\frac{3}{5}\right)^2\left(\frac{2}{5}\right) = \frac{36}{125} + \frac{54}{125} = \frac{90}{125}$.
$B \cap C$ is the event $\{X = 2\}$, which has probability $\binom{3}{2}\left(\frac{3}{5}\right)^2\left(\frac{2}{5}\right) = \frac{54}{125}$.
B and C are not independent since

$$\mathbb{P}(B \cap C) = \frac{54}{125} \neq \frac{1458}{3125} = \frac{81}{125} \cdot \frac{90}{125} = \mathbb{P}(B) \cdot \mathbb{P}(C).$$

(e) $Y \sim \text{Hypergeom}(20, 9, 6)$. Its probability mass function is given by

$$\mathbb{P}(Y = k) = \frac{\binom{9}{k}\binom{11}{6-k}}{\binom{20}{6}}, \qquad k = 0, \dots, 6.$$

(f) Using the r.v. Y, the desired probability is equal to

$$\mathbb{P}(Y \geq 5) = \mathbb{P}(Y = 6) + \mathbb{P}(Y = 5) = \frac{\binom{9}{6}\binom{11}{0}}{\binom{20}{6}} + \frac{\binom{9}{5}\binom{11}{1}}{\binom{20}{6}} = \frac{7}{3230} + \frac{231}{6460} = \frac{49}{1292} \approx 0.0379.$$