Mock Short Test 2

Time available: 45 minutes

Exercise 1

Let X be a continuous random variable with density function

$$f_X(x) = \begin{cases} 2(x-1) & \text{if } 1 \leq x \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Find the cumulative distribution function $F_X(x)$ of the random variable X for all $x \in \mathbb{R}$

Solution

The CDF F_X can be computed in any $x \in \mathbb{R}$ by integrating the density as follows

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt.$$

For any $x < 1$, the density f is constantly equal to zero, so we trivially have $F_X(x) = \int_{-\infty}^{x} 0\, dt = 0$. For $1 \leq x \leq 2$, the CDF takes value

$$F_X(x) = \int_{-\infty}^{1} 0\, dt + \int_{1}^{x} 2(t-1)\, dt = \left[t^2 - 2t\right]_1^x = x^2 - 2x + 1 = (x-1)^2$$

Since the density is constantly equal to zero again after $x = 2$, the value of the CDF stays constant as well and we have $F_X(x) = 1$ for any $x > 2$.

Exercise 2

Consider an urn with 3 red and 7 blue balls.

(a) Mary takes 2 arbitrary balls from this urn, *without replacement*. Let X be the number of blue balls that she obtains. Give the probability mass function of X and the name of its distribution.

(b) We put the balls that Mary took back, so we are back with an urn with 3 red and 7 blue balls. We now let Boris take a ball from this urn five times, *with replacement*. Let Y be the total number of times Boris obtains a blue ball. Give the probability mass function and the name of the distribution of Y.

(c) Finally, Mo takes balls from the same urn with 3 red and 7 blue balls, *with replacement*, and continues until he takes a red ball. Let W be the number of balls Mo needs to draw until he succeeds. Give the probability mass function and the name of the distribution of W.

Solution

(a) X is a hypergeometric distribution with parameters. The probability mass function of X is

$$\mathbb{P}(X=0) = \frac{\binom{7}{0}\binom{3}{2}}{\binom{10}{2}} = \frac{1}{15}, \qquad \mathbb{P}(X=1) = \frac{\binom{7}{1}\binom{3}{1}}{\binom{10}{2}} = \frac{7}{15}, \qquad \mathbb{P}(X=2) = \frac{\binom{7}{2}\binom{3}{0}}{\binom{10}{2}} = \frac{7}{15}.$$

(b) The probability mass function of Y is $\mathbb{P}(Y=y) = \binom{5}{y}\left(\frac{7}{10}\right)^y\left(\frac{3}{10}\right)^{5-y}$ for every $y = 0, 1, 2, 3, 4, 5$. Y has the binomial distribution with parameters 5 and $\frac{7}{10}$, i.e., $Y \sim \text{Bin}\left(5, \frac{7}{10}\right)$.

(c) The probability mass function of W is $\mathbb{P}(W=w) = \left(\frac{7}{10}\right)^{w-1}\frac{3}{10}$ for every $w = 1, 2, \ldots$. W has the geometric distribution with parameter $\frac{3}{10}$, i.e., $W \sim \text{Geom}\left(\frac{3}{10}\right)$.

Exercise 3

Suppose we flip a fair coin 3 times. Let A be the event that we obtain heads at most once and let B be the event that we obtain heads at least once and tails at least once. Are A and B independent?

Solution

We can calculate

$$\mathbb{P}(A) = \mathbb{P}(3 \text{ tails}) + \mathbb{P}(1 \text{ heads and } 2 \text{ tails}) = \left(\frac{1}{2}\right)^3 + \binom{3}{1}\left(\frac{1}{2}\right)^3 = \frac{1}{2}$$

and

$$\mathbb{P}(B) = \mathbb{P}(2 \text{ heads and } 1 \text{ tails}) + \mathbb{P}(1 \text{ heads and } 2 \text{ tails}) = \binom{3}{1}\left(\frac{1}{2}\right)^3 + \binom{3}{1}\left(\frac{1}{2}\right)^3 = \frac{3}{4}$$

Alternatively, $\mathbb{P}(B) = 1 - \mathbb{P}(3 \text{ heads}) - \mathbb{P}(3 \text{ tails}) = 1 - \left(\frac{1}{2}\right)^3 - \left(\frac{1}{2}\right)^3 = \frac{3}{4}$.

Further,

$$\mathbb{P}(A \cap B) = \mathbb{P}(1 \text{ heads and } 2 \text{ tails}) = \binom{3}{1}\left(\frac{1}{2}\right)^3 = \frac{3}{8}.$$

Therefore, A and B are independent, since

$$\mathbb{P}(A \cap B) = \frac{3}{8} = \frac{1}{2} \cdot \frac{3}{4} = \mathbb{P}(A) \cdot \mathbb{P}(B).$$

Exercise 4

We have an urn with 3 red balls and 2 blue balls. We pick a sample of three balls *without replacement* and put these balls in a second urn that was previously empty. Then we sample three balls from the second urn *with replacement*. Let X be the number of blue balls taken from the first urn and let Y be the number of blue balls taken from the second urn. Compute $\mathbb{P}(X = 1 \mid Y = 2)$.

Solution

We have

$$\mathbb{P}(X = 0) = \frac{\binom{3}{3}\binom{2}{0}}{\binom{5}{3}} = \frac{1}{10}, \qquad \mathbb{P}(X = 1) = \frac{\binom{3}{2}\binom{2}{1}}{\binom{5}{3}} = \frac{6}{10}, \qquad \mathbb{P}(X = 2) = \frac{\binom{3}{1}\binom{2}{2}}{\binom{5}{3}} = \frac{3}{10}.$$

Further,

$$\begin{aligned}
\mathbb{P}(X = 1 \mid Y = 2) &= \frac{\mathbb{P}(X = 1 \cap Y = 2)}{\mathbb{P}(Y = 2)} \\
&= \frac{\mathbb{P}(Y = 2 \mid X = 1)\mathbb{P}(X = 1)}{\mathbb{P}(Y = 2 \mid X = 0)\mathbb{P}(X = 0) + \mathbb{P}(Y = 2 \mid X = 1)\mathbb{P}(X = 1) + \mathbb{P}(Y = 2 \mid X = 2)\mathbb{P}(X = 2)} \\
&= \frac{\frac{6}{10}\binom{3}{1}\left(\frac{1}{3}\right)^2\frac{2}{3}}{\frac{1}{10}\cdot 0 + \frac{6}{10}\binom{3}{1}\left(\frac{1}{3}\right)^2\frac{2}{3} + \frac{3}{10}\binom{3}{2}\left(\frac{2}{3}\right)^2\frac{1}{3}} = \cdots = \frac{1}{2}.
\end{aligned}$$