Short Test 1 Solutions – Probability Theory course 2023

Friday February 24th, 16:30–17:15 (45 minutes)

Full name: **Student number:** **TA name:**

- Write your full name and student number on both sheets provided, together with the name of your TA.
- **This pair of sheets is the only one that you need to hand in to your TA at the end of the test**. Please write your answers *starting with the back of this sheet*. If you run out of space, ask the TA for an extra sheet and do not forget to also write there your name and student number. Scrap paper is provided upon request. Any solution written on scrap paper will not be graded, make sure to copy it on this sheet.
- **No material of any kind is allowed during the short test**. In particular, no book and no notes.
- You may use a **simple calculator** for this test, but this is not necessary. In fact, you do not have to work out powers, fractions, binomial coefficients, products, etc. Smartphones are not allowed.
- Use proper notation and terminology, justify in a clear way all your steps, and do not just write the final numerical answer.
- You have 45 minutes available for the test. You cannot leave the short test before 17:00. If you want to leave before the end of the test, raise your hand, and wait for the TA to collect your sheet.
- Your short test grade is given by 1+ the number of points (thus 10 is the maximum), rounded to the 1st decimal.

Exercise 1 [3 points]

A fair die with six faces numbered from one to six is thrown twice.

(a) [0.5 points] Define the probability space in full.

(b) [0.5 points] What is the probability that a six turns up exactly once?

(c) [1 point] What is the probability that the sum of the scores is 4?

(d) [1 point] What is the probability that the sum of the scores is divisible by 3?

Solution

(a) The sample space is $\Omega = \{\text{ordered pairs } (i,j) \; : \; i,j \in \{1,\dots,6\}\}$, the set of events is the power set of Ω, i.e., $\mathcal{F} = 2^\Omega$, and since we have a fair die all possible outcomes are equiprobable $\mathbb{P}((i,j)) = \frac{1}{|\Omega|} = \frac{1}{36}$.

(b) Let U the event that exactly one six turns up. There are exactly 10 outcomes that have exactly once the number six, namely $U = \{(1,6),(2,6),(3,6),(4,6),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5)\}$. Hence, $\mathbb{P}(U) = \frac{|U|}{|\Omega|} = \frac{10}{36} = \frac{5}{18}$.

(c) The event we are interested is $\{S = 4\} = \{(1,3),(2,2),(3,1)\}$, so $\mathbb{P}(S = 4) = \frac{|\{S=4\}|}{|\Omega|} = \frac{3}{36}$. Alterantively, using the addivitivity property, we could calculate $\mathbb{P}(S = 4) = \mathbb{P}((1,3)) + \mathbb{P}((2,2)) + \mathbb{P}((3,1)) = \frac{3}{36} = \frac{4}{12}$.

(d) The event we are interested is $\{S \equiv 0 \pmod 3\} = \{S = 3\} \cup \{S = 6\} \cup \{S = 9\} \cup \{S = 12\}$. By enumerating their elements, we can check these four events consists of 2, 5, 4, 1 outcomes. Either by calculating the cardinality of $\{S \equiv 0 \pmod 3\}$ or using the additive property, we get

$$\mathbb{P}(S \equiv 0 \pmod 3) = \mathbb{P}(S = 3) + \mathbb{P}(S = 6) + \mathbb{P}(S = 9) + \mathbb{P}(S = 12) = \frac{2}{36} + \frac{5}{36} + \frac{4}{36} + \frac{1}{36} = \frac{1}{3}.$$

Exercise 2 [2 points]

Let A and B be events with probabilities $\mathbb{P}(A) = \frac{3}{4}$ and $\mathbb{P}(B) = \frac{1}{3}$.

(a) [1 point] Justifying every step, show that $\frac{1}{12} \leq \mathbb{P}(A \cap B) \leq \frac{1}{3}$.

(b) [1 point] Justifying every step, find the best upper bound and a lower bound for $\mathbb{P}(A \cup B)$.

Solution

(a) For the lower bound, using first the inclusion-exclusion formula and then the monotonicity of $\mathbb{P}$, we get $\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1 = \frac{1}{12}$. For the upper bound, since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, by the monotonicity of $\mathbb{P}$, we obtain $\mathbb{P}(A \cap B) \leq \min\{\mathbb{P}(A), \mathbb{P}(B)\} = \frac{1}{3}$.

(b) Consider again the inclusion-exclusion formula where we make explicit the probability of $A \cup B$, namely

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

Since $\mathbb{P}(\cdot)$ is a probability measure $-\mathbb{P}(A \cap B) \leq 0$ and thus, $\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B)$. At the same time, any probability cannot take values larger than 1, so we have $\mathbb{P}(A \cup B) \leq \min\{\mathbb{P}(A) + \mathbb{P}(B), 1\} = 1$. Lastly, since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, by monotonicity of $\mathbb{P}$, we obtain $\mathbb{P}(A \cup B) \geq \max\{\mathbb{P}(A), \mathbb{P}(B)\} = \frac{3}{4}$.

Exercise 3 [2 points]

Assume that each newborn child is equally likely to be a boy or a girl. Consider a family that has two children.

(a) [1 point] What is the probability that both are girls given that the oldest is a girl?

(b) [1 point] What is the probability that both are girls given that at least one is a girl?

Solution

We implicitly use as sample space $\Omega = \{(G, G), (G, B), (B, G), (B, B)\}$ (the pairs are ordered, first in the vector is the younger siblings). All these pairs of children are equiprobable, each with probability $1/4$.

Let $B = \{\text{both are girls}\} = \{(G, G)\}$, $E = \{\text{oldest is girl}\} = \{(G, G), (B, G)\}$, $L = \{\text{at least one is a girl}\} = \{(G, G), (G, B), (B, G)\}$; and $L^c = \{\text{no girls}\} = \{(B, B)\}$.

(a) Using first the definition of conditional probability and then the fact that $E \subset B$, we get

$$\mathbb{P}(B|E) = \frac{\mathbb{P}(B \cap E)}{\mathbb{P}(E)} = \frac{\mathbb{P}(B)}{\mathbb{P}(E)} = \frac{1/4}{1/2} = \frac{1}{2}.$$

(b) $\mathbb{P}(L) = 1 - \mathbb{P}(L^c) = 1 - \mathbb{P}(\text{no girls}) = 1 - \dfrac{1}{4}$, so that $\mathbb{P}(B|L) = \dfrac{\mathbb{P}(B \cap L)}{\mathbb{P}(L)} = \dfrac{\mathbb{P}(B)}{\mathbb{P}(L)} = \dfrac{1/4}{3/4} = \dfrac{1}{3}$.

Exercise 4 [2 points]

Urn 1 contains two white balls and one black ball, while urn 2 contains one white ball and five black balls. One ball is drawn at random from urn 1 and placed in urn 2. A ball is then drawn from urn 2. It happens to be white. What is the probability that the transferred ball was white?

Solution

Let $T_W = \{\text{transferred ball is white}\}$; $T_B = \{\text{transferred ball is black}\}$; $W = \{\text{white ball is drawn from urn 2}\}$. Using Bayes' formula, we can calculate

$$\mathbb{P}(T_W|W) = \frac{\mathbb{P}(W|T_W)P(T_W)}{\mathbb{P}(W|T_W)P(T_W) + \mathbb{P}(W|T_B)P(T_B)} = \frac{\frac{2}{7} \cdot \frac{2}{3}}{\frac{2}{7} \cdot \frac{2}{3} + \frac{1}{7} \cdot \frac{1}{3}} = \frac{4}{5}.$$