Mock Short Test 1 – Solutions

Time available: 45 minutes

Full name: **Student number:** **TA name:**

- Write your full name and student number on both sheets provided, together with the name of your TA.
- **This pair of sheets is the only one that you need to hand in to your TA at the end of the test**. Please write your answers *starting with the back of this sheet*. If you run out of space, ask the TA for an extra sheet and do not forget to also write there your name and student number.
- **No material of any kind is allowed during the short test**. In particular, no book and no notes.
- Scrap paper is provided upon request. Any solution written on scrap paper will not be graded, make sure to copy it on this sheet.
- You may use a **simple calculator** for this test, but this is not necessary. In fact, you do not have to work out powers, fractions, binomial coefficients, products, etc. Phones are not allowed (and so is their built-in calculator).
- Use proper notation and terminology, justify all your steps, explain your answers clearly, and do not just write the numerical answer.
- You have 45 minutes available for the test. You cannot leave the short test before 17:00. If you want to leave after 17:00 but before the end of the test, raise your hand, and wait for the TA to collect your sheet.
- Your short test grade is given by 1+ the number of points (thus 10 is the maximum), rounded to the 1st decimal.

Exercise 1 [2 points]

Prof Z. has two drawers with socks. The first drawer contains 4 white socks and 3 black socks. The second drawer contains 5 white socks and 2 black socks. Prof Z. first chooses a drawer at random in such a way that the first drawer is chosen with a probability of 2/3 and the second drawer is chosen with a probability of 1/3. Then, he randomly draws (without replacement) three socks from the chosen drawer.

(a) Determine the probability that he obtains at least two black socks.

(b) Calculate the conditional probability that the first drawer was chosen given the event that all three drawn socks have the same color.

Solution

(a) Using the partition $\Omega = \{\text{drawer 1}\} \cup \{\text{drawer 2}\}$ and the definition of conditional probability, we get

$$\begin{aligned}
\mathbb{P}(\text{at least 2 black socks}) &= \mathbb{P}(\{\text{at least 2 black socks}\} \cap \{\text{drawer 1}\}) + \mathbb{P}(\{\text{at least 2 black socks}\} \cap \{\text{drawer 2}\}) \\
&= \mathbb{P}(\text{at least 2 black socks} \mid \text{drawer 1}) \cdot \mathbb{P}(\text{drawer 1}) \\
&\quad + \mathbb{P}(\text{at least 2 black socks} \mid \text{drawer 2}) \cdot \mathbb{P}(\text{drawer 2}) \\
&= \mathbb{P}(\text{drawer 1}) \cdot (\mathbb{P}(\text{2 black and 1 white} \mid \text{drawer 1}) + \mathbb{P}(\text{3 black} \mid \text{drawer 1})) \\
&\quad + \mathbb{P}(\text{drawer 2})\mathbb{P}(\text{2 black and 1 white} \mid \text{drawer 2}) \\
&= \frac{2}{3}\left(\frac{\binom{3}{2}\binom{4}{1}}{\binom{7}{3}} + \frac{\binom{3}{3}}{\binom{7}{3}}\right) + \frac{1}{3}\frac{\binom{2}{2}\binom{5}{1}}{\binom{7}{3}} = \frac{31}{105}.
\end{aligned}$$

(b) Using Bayes' formula,

$$\begin{aligned}
&\mathbb{P}(\text{drawer 1} \mid \text{all same color}) \\
&= \frac{\mathbb{P}(\text{all same color} \mid \text{drawer 1}) \cdot \mathbb{P}(\text{drawer 1})}{\mathbb{P}(\text{all same color} \mid \text{drawer 1}) \cdot \mathbb{P}(\text{drawer 1}) + \mathbb{P}(\text{all same color} \mid \text{drawer 2}) \cdot \mathbb{P}(\text{drawer 2})} \\
&= \frac{\frac{2}{3}\frac{\binom{4}{3}+\binom{3}{3}}{\binom{7}{3}}}{\frac{2}{3}\frac{\binom{4}{3}+\binom{3}{3}}{\binom{7}{3}} + \frac{1}{3}\frac{\binom{5}{3}}{\binom{7}{3}}} = \cdots = \frac{1}{2}.
\end{aligned}$$

Exercise 2 [2 points]
Suppose that A and B are two events such that $\mathbb{P}(A) = 0.4$, $\mathbb{P}(B) = 0.3$, and $\mathbb{P}(A \cup B) = 0.5$.

(a) Are A and B independent?

(b) Compute $\mathbb{P}(A^c \cap B^c)$.

Solution

(a) We first calculate $\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B) = 0.3 + 0.4 - 0.5 = 0.2$. A and B are not independent since $\mathbb{P}(A \cap B) = 0.2 \neq 0.4 \cdot 0.3 = \mathbb{P}(A)\mathbb{P}(B)$.
(b) Using the relationship between the probability of an event and that of its complement, we get $\mathbb{P}(A^c \cap B^c) = 1 - \mathbb{P}((A^c \cap B^c)^c) = 1 - \mathbb{P}(A \cup B) = 1 - 0.5 = 0.5$.

Exercise 3 [3 points]
A fair die is thrown five times. Let A be the event that '4' occurs at least once and let B be the event that the outcome '2' occurs exactly 3 times.

(a) Compute $\mathbb{P}(A)$ and $\mathbb{P}(B)$.

(b) Compute $\mathbb{P}(A \cup B)$.

Solution

(a) The sample space is $\Omega = \{(x_1, \ldots, x_5) \ : \ x_1, \ldots, x_5 \in \{1, \ldots, 6\}\}$ and $|\Omega| = 6^5$. Then,

$$\mathbb{P}(A) = 1 - \mathbb{P}(A^c) = 1 - \frac{|A^c|}{|\Omega|} = 1 - \frac{5^5}{6^5} \qquad \text{and} \qquad \mathbb{P}(B) = \frac{\binom{5}{3} \cdot 5^2}{6^5}.$$

(b) The desired probability can be calculated as $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$. The first two terms have already been calculated in (a), so we only need to calculate $\mathbb{P}(A \cap B)$. To do so, we decompose the event $A \cap B$ using the partition of Ω based on the rolled number of '4's.

$$\begin{aligned}\mathbb{P}(A \cap B) &= \mathbb{P}(\text{three times '2', one time '4', another number different from '2' and '4'}) \\ &\quad + \mathbb{P}(\text{three times '2' and two times '4'}) \\ &= \frac{\binom{5}{3} \cdot 2 \cdot 4}{6^5} + \frac{\binom{5}{3}}{6^5}\end{aligned}$$

Exercise 4 [2 points]
An urn contains 4 red balls and 3 white balls. Take 3 balls from the urn, without replacement, and one by one (so with order). Let X be the total number of white balls obtained and let W_1 be the event that the first ball is white. Compute $\mathbb{P}(W_1 \mid X = 2)$.

Solution

All outcomes are equally likely and $|\Omega| = 7 \cdot 6 \cdot 5$. Let W_i be the event that the i-th ball is white and R_i the event that the i-th ball is red. Then,

$$\begin{aligned}\mathbb{P}(W_1 \mid X = 2) &= \frac{\mathbb{P}(W_1 \cap X = 2)}{\mathbb{P}(X = 2)} = \frac{\mathbb{P}(W_1 W_2 R_3) + \mathbb{P}(W_1 R_2 W_3)}{\mathbb{P}(W_1 W_2 R_3) + \mathbb{P}(W_1 R_2 W_3) + \mathbb{P}(R_1 W_2 W_3)} \\ &= \frac{\frac{3 \cdot 2 \cdot 4}{7 \cdot 6 \cdot 5} + \frac{3 \cdot 4 \cdot 2}{7 \cdot 6 \cdot 5}}{\frac{3 \cdot 2 \cdot 4}{7 \cdot 6 \cdot 5} + \frac{3 \cdot 4 \cdot 2}{7 \cdot 6 \cdot 5} + \frac{4 \cdot 3 \cdot 2}{7 \cdot 6 \cdot 5}} = \frac{2}{3}\end{aligned}$$

All the terms appearing above have been computed using directed counting. Alternatively, we could have used the properties of conditional probabilities, e.g.,

$$\mathbb{P}(W_1 W_2 R_3) = \mathbb{P}(W_1)\mathbb{P}(W_2 \mid W_1)\mathbb{P}(R_3 \mid W_1 W_2) = \frac{3}{7} \cdot \frac{2}{6} \cdot \frac{4}{5}.$$