Final Exam Probability Theory

July 7, 2022

- You can score 36 points. Your grade is given by (4 + number of points)/4.
- You may use a simple calculator, but it is not allowed to use a graphical or a programmable calculator.
- Explain your answers clearly!

Question 1. Dick possesses 4 blue cups, 3 green cups and 2 white cups. Cups of the same colour are indistinguishable. Het puts the cups on a shelve, in some arbitrary order.
(a) (3 points) How many different orders of the 9 coloured cups exist?
(b) (3 points) What is the probability that the cups are placed in such a way that all cups are grouped by colour?
(c) (2 points) What is the probability that the first three cups, counted from the left, have the same colour?
(d) (3 points) Now let the stochastic variable X be defined as the maximal number of cups of the same colour that are grouped together. Compute $P(X = 4)$.

Question 2. (4 points) We throw a dice twice. The outcome of the first throw is called X, and the outcome of the second is called Y. Finally, we define the random variable Z as $Z = 0$ if $X + Y$ is even, and $Z = 1$ if $X + Y$ is odd. Are X and Z independent? (Hint: recall for yourself the definition of independence for random variables.)

Question 3. Joe and Maria throw a die, at the same time. If Joe's outcome is at least as high as Maria's, then they stop. Otherwise they repeat the experiment until the first time that Joe's outcome is at least as high as Maria's, but not beyond the fifth experiment. That is, in case they get to a fifth experiment, they stop after the fifth experiment regardless of the outcome of the fifth experiment. The total number of times that Joe (and

hence also Maria) throws his die is denoted by X.
(a) (2 points) Compute the probability mass function of X.
(b) (4 points) Compute the expectation and variance of X.

Question 4. (3 points) You have four coins that look identical. However, it is known that the coins do differ from each in the following respect: two of them are fair, one is unfair with probability $2/3$ of seeing heads, and one is unfair with probability $1/3$ of seeing heads. Because the coins look identical, you can not immdiately tell which coins are fair and which ones are unfair. You pick a coin at random, toss it three times, and on all three tosses heads come up. Given this event, compute the conditional probability that the coin you picked is the one with probability $2/3$ for seeing heads.

Question 5. Let (X, Y) have joint density function $f(x, y) = xe^{-x(y+1)}$, for $x, y > 0$ and $f(x, y) = 0$ elsewhere.
(a) (3 points) Compute the marginal densities of X and Y.
(b) (3 points) Compute the conditional density function $f_{Y|X}(y|x)$. Which common distribution is this? Specify the name of the distribution and the parameter value.
(c) (3 points) Compute $E(Y|X = x)$. Provide a full calculation; a reference to a known distribution will not suffice.

Question 6. (3 points) Let X and Y be independent and uniformly distributed on $(0, 1)$. Compute the density function of $X - Y$.