Question 1

a) $$\frac{9!}{4!\,3!\,2!}$$

b) $$\frac{3!}{\dfrac{9!}{4!\,3!\,2!}}$$

$3!$ → the number of possible sequences of the 3 groups/colors

$\dfrac{9!}{4!\,3!\,2!}$ } answer from a)

c) P(first 3 cups from the left are blue)+

+P(first 3 cups from the left are green)=

$$= \frac{4}{9}\cdot\frac{3}{8}\cdot\frac{2}{7} + \frac{3}{9}\cdot\frac{2}{8}\cdot\frac{1}{7}$$

d) $P(X=4) =$ P(all blue cups are grouped together)=

$$= \frac{6 \cdot \dfrac{5!}{3!\cdot 2!}}{\dfrac{9!}{4!\,3!\,2!}}$$

$\dfrac{5!}{3!\cdot 2!}$ } → number of possible orders for the remaining (~~green~~white cups)

6 → number of the possible solutions for the 1st blue cup

Question 2

$P(Z=0) = \frac{18}{36} = P(Z=1)$ and $P(X=i) = \frac{1}{6}$ for $i=1,\dots,6$

For examaple, $P(Z=0, X=1) = P(\{(1,1),(1,3),(1,5)\}) = \frac{3}{36} = \frac{1}{2}\cdot\frac{1}{6} = P(Z=0)\cdot P(X=1)$

In general $P(Z=0, X=i) = P(X=i$ and 3 options for Y to make $X+Y$ even$)=$

$$= \frac{3}{36} = \frac{1}{12} = \frac{1}{2}\cdot\frac{1}{6} = P(Z=0)\,P(X=i)$$

Similarly $P(Z=1, X=i) = P(X=i$ and 3 options for Y to make $X+Y$ odd$)=$

$$= \frac{3}{36} = \frac{1}{12} = \frac{1}{2}\cdot\frac{1}{6} = P(Z=1)\cdot P(X=i)$$

(or $P(Z=1, X=i) = P(X=i) - P(Z=0, X=i) = \frac{1}{6} - \frac{3}{36} = \frac{3}{36} = \frac{1}{12} = \frac{1}{2}\cdot\frac{1}{6} =$

$$= P(Z=0)\cdot P(X=i)$$

In conclusion, the definition of independence for X and Z indeed holds.

Question 3

a) With $p = P$(Joe's outcome is no less than Maria's) =

$$= \frac{1}{6}\cdot\frac{6}{6}+\frac{1}{6}\cdot\frac{5}{6}+\frac{1}{6}\cdot\frac{4}{6}+\frac{1}{6}\cdot\frac{3}{6}+\frac{1}{6}\cdot\frac{2}{6}+\frac{1}{6}\cdot\frac{1}{6} = \frac{21}{36}$$

(This is by conditioning on Maria's outcome, another solution is that 21 out of the 36 outcomes fit)

$f(i) = (1-p)^{i-1}p$, for $i = 1,2,3,4$

and $f(5) = 1 - (f(1)+f(2)+f(3)+f(4))$

b) $EX = 1\cdot f(1) + 2\cdot f(2) + 3\cdot f(3) + 4\cdot f(4) + 5\cdot f(5) =$

$$= \left(\frac{15}{36}\right)^0\cdot\frac{21}{36} + 2\cdot\left(\frac{15}{36}\right)^1\frac{21}{36} + 3\left(\frac{15}{36}\right)^2\cdot\frac{21}{36} + 4\left(\frac{15}{36}\right)^3\cdot\frac{21}{36} + 5\left(\frac{15}{36}\right)^4\cdot\frac{21}{36} \simeq 1.63$$

$E(X^2) = 1^2\cdot f(1) + 2^2\cdot f(2) + 3^2\cdot f(3) + 4^2 f(4) + 5^2 f(5) =$

$$= \frac{21}{36} + 4\cdot\frac{15}{36}\cdot\frac{21}{36} + 9\left(\frac{15}{36}\right)^2\cdot\frac{21}{36} + 16\left(\frac{15}{36}\right)^3\cdot\frac{21}{36} + 25\left(\frac{15}{36}\right)^4\cdot\frac{21}{36} \simeq 3.56$$

$Var(X) = E(X^2) - [E(X)]^2 \simeq 0.92$

Question 4

C_1: fair coin, $P_{heads} = \frac{1}{2}$

C_2: fair coin, $P_{heads} = \frac{1}{2}$

C_3: unfair coin, $P_{heads} = \frac{2}{3}$

C_4: unfair coin, $P_{heads} = \frac{1}{3}$

$$P(C_3/HHH) = \frac{P(HHH \cap C_3)}{P(HHH)} = \frac{P(HHH/C_3)\cdot P(C_3)}{P(HHH/C_1)\cdot P(C_1) + P(HHH/C_2)\cdot P(C_2) + P(HHH/C_3)\cdot P(C_3) + P(HHH/C_4)\cdot P(C_4)} =$$

$$= \frac{\left(\frac{2}{3}\right)^3 \cdot \frac{1}{4}}{\frac{1}{4}\cdot\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3\cdot\frac{1}{4} + \left(\frac{2}{3}\right)^3\cdot\frac{1}{4} + \left(\frac{1}{3}\right)^3\cdot\frac{1}{4}} = \frac{\frac{8}{27}}{2\cdot\frac{1}{8} + \frac{8}{27} + \frac{1}{27}} = \frac{32}{63}$$

Question 5

a) $f_X(x) = \int_0^\infty x e^{-x(y+1)}\,dy = xe^{-x}\int_0^\infty e^{-xy}\,dy = xe^{-x}\left[-\frac{1}{x}e^{-xy}\right]_0^\infty = xe^{-x}\left[0+\frac{1}{x}\right] = e^{-x},\ x>0$

$$f_X(x) = \begin{cases} e^{-x}, & x>0 \\ 0, & x\le 0 \end{cases}$$

$f_Y(y) = \int_0^\infty x e^{-x(y+1)}\,dx = \left[-\frac{1}{y+1}e^{-x(y+1)}\cdot x\right]_0^\infty + \int_0^\infty \frac{1}{y+1}e^{-x(y+1)}\,dx =$

$= 0 + \left[-\frac{1}{(y+1)^2}e^{-x(y+1)}\right]_0^\infty = \frac{1}{(y+1)^2},\ y>0$

$$f_Y(y) = \begin{cases} \frac{1}{(y+1)^2}, & y>0 \\ 0, & y\le 0 \end{cases}$$

b) $f_{Y/X}(y/x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{xe^{-x(y+1)}}{e^{-x}} = xe^{-xy},\ y>0$

this is an exponential (x) distribution

c) $E(Y/X=x) = \int_{-\infty}^{\infty} y\cdot f_{Y/X}(y/x)\,dy = \int_0^\infty y\cdot x e^{-xy}\,dy =$

$= \left[\frac{-y\cdot x}{x}e^{-xy}\right]_0^\infty + \int_0^\infty \frac{x}{x}e^{-xy}\,dy = 0 + \left[-\frac{1}{x}e^{-xy}\right]_0^\infty = 0+\frac{1}{x} = \frac{1}{x},\ x>0$

Question 6

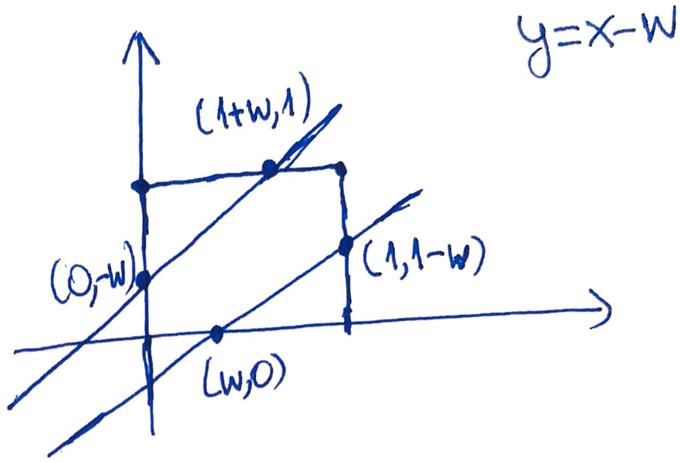
X and Y Independent and uniformly distributed on $(0,1)$

so $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$

$f_X(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{else} \end{cases}$, $f_Y(y) = \begin{cases} 1, & 0 \le y \le 1 \\ 0, & \text{else} \end{cases}$

$f_{X,Y}(x,y) = \begin{cases} 1, & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0, & \text{else} \end{cases}$

For $Z = X - Y$



area

$\rightarrow -1 \le w \le 0: F_W(w) = \text{Area} = \frac{1}{2}(1-(-w)) \cdot (1+w) = \frac{1}{2}(1+w)^2$

area

$\rightarrow 0 < w \le 1: F_W(w) = \text{Area} = 1 - \frac{1}{2}(1-w)(1-w) = 1 - \frac{1}{2}(1-w)^2$

thus $F_W(w) = \begin{cases} 0, & w < -1 \\ \frac{1}{2}(1+w)^2, & -1 \le w \le 0 \\ 1 - \frac{1}{2}(1-w)^2, & 0 < w \le 1 \\ 1, & w > 1 \end{cases}$

And now differentiating the cdf we get:

$f_W(w) = \begin{cases} 1+w, & -1 \le w \le 0 \\ 1-w, & 0 < w \le 1 \\ 0, & \text{else} \end{cases}$