Final Exam Probability Theory

June 2, 2022, 15:30-17:30

- Every item is worth 3 points, so you can score 36 points. Your grade is given by (4 + number of points)/4.
- You may use a simple calculator, but it is not allowed to use a graphical or a programmable calculator.
- Explain your answers clearly!

Question 1. Let X and Y be jointly continuous random variables with joint density function

$$f_{X,Y}(x,y) = \frac{1}{x},\ 0 < y \leq x \leq 1,$$

and $f_{X,Y}(x,y) = 0$ elsewhere.
(a) Show that X has a uniform distribution on the interval $(0,1]$.
(b) Compute the marginal density $f_Y(y)$ of Y.
(c) Show that the conditional density function $f_{Y|X}(y|x)$ is equal to $1/x$, if $0 < y \leq x \leq 1$, and 0 otherwise.
(d) Use (a) and (c) to compute $E(Y)$.

Question 2. Let Z have a standard normal distribution and let $X := Z^2$.
(a) Show that, for $0 \leq x_1 \leq x_2$,

$$P(x_1 \leq X \leq x_2) = 2\Phi(\sqrt{x_2}) - 2\Phi(\sqrt{x_1}),$$

where Φ denote the distribution function of Z.
(b) Compute the density function of X.

Question 3. A chicken produces N eggs, where N is a random variable with a Poisson distribution with parameter λ. Each of the eggs hatches with probability p, independently of the other eggs. So, if K denotes the number of chicks, we have

$$p_{K|N}(k|n) = \binom{n}{k} p^k (1-p)^{n-k}.$$

(a) Show that $E(K|N = n) = pn$.
(b) Use (a) to show that $E(K) = p\lambda$.

Question 4. It is conjectured that one can see from a person's handwriting, whether this person is a man or a woman. Someone inspects 1500 examples of handwriting. Suppose he makes a random guess each time (i.e., guesses correctly with probability 1/2), and that his assessments are independent of each other. We denote the number of correct guesses out of the 1500 examples by X.

Give an approximation of $P(X \geq 950)$ in terms of Φ (the distribution function of the standard normal distribution).

Question 5. Let X be a uniform random variable on $[0, 1]$ and Y a uniform random variable on $[3, 4]$.
(a) Supposing that X and Y are independent, find the cumulative distribution function $F_W(w)$ of $W := Y - X$.
Hint: it is convenient to distinguish between several cases for w, including $2 < w \leq 3$ and $3 < w \leq 4$.
(b) Compute the variance of W from (a).
(c) Suppose now that X and Y are related as $Y = X + 3$. Compute the variance of $Y - X$ in this case.