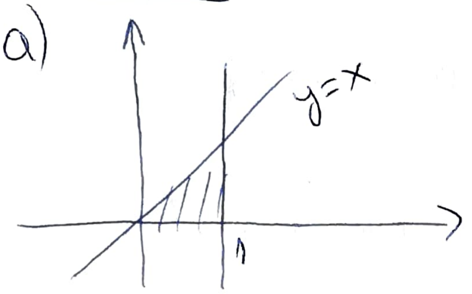
Question 1

a)



$$f_X(x)=\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

for $0 \leq x \leq 1$

$$f_X(x)=\int_0^x \frac{1}{x}\,dy=\left[\frac{y}{x}\right]_0^x=1$$

Thus, $f_X(x)=\begin{cases}1, & \text{if } x\in[0,1]\\ 0, & \text{if } x\notin[0,1]\end{cases}$

So, we proved X has a uniform distribution on $[0,1]$

b) $f_Y(y)=\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$

for $0\leq y\leq 1$: $f_Y(y)=\int_y^1 \frac{1}{x}\,dx=\left[\ln x\right]_y^1=-\ln y$

Thus, $f_Y(y)=\begin{cases}-\ln y, & \text{if } 0\leq y\leq 1\\ 0, & \text{otherwise}\end{cases}$

c) $f_{Y/X}(y/x)=\dfrac{f_{X,Y}(x,y)}{f_X(x)}=\dfrac{\frac{1}{x}}{1}=\dfrac{1}{x}$

$f_{Y/X}(y/x)=\begin{cases}\frac{1}{x}, & \text{for } 0<y\leq x\leq 1\\ 0, & \text{otherwise}\end{cases}$

d) $E(Y)=\int_{-\infty}^{\infty} E(Y/X=x)\,f_X(x)\,dx$

$E(Y/X=x)=\int_{-\infty}^{\infty} y\cdot f_{Y/X}(y/x)\,dy=\int_0^x y\cdot\frac{1}{x}\,dy=\left[\frac{y^2}{2x}\right]_0^x=\frac{x}{2}, \quad 0<x\leq 1$

$E(Y)=\int_0^1 \frac{x}{2}\cdot 1\,dx=\left[\frac{x^2}{4}\right]_0^1=\frac{1}{4}$

Question 2

(if $x_1>0$, because event on the left is a disjoint union of events on the right, if $x_1=0$ also true because $P(Z=0)=0$)

a) $P(x_1 \le X \le x_2) = P(x_1 \le Z^2 \le x_2) \overset{*}{=} P(\sqrt{x_1} \le Z \le \sqrt{x_2}) + P(\sqrt{x_1} \le -Z \le \sqrt{x_2}) =$

(for $Z \ge 0$) (for $Z<0$)

$= \Phi(\sqrt{x_2}) - \Phi(\sqrt{x_1}) + P(-\sqrt{x_1} \ge Z \ge -\sqrt{x_2}) =$

$= \Phi(\sqrt{x_2}) - \Phi(\sqrt{x_1}) + \Phi(-\sqrt{x_1}) - \Phi(-\sqrt{x_2}) =$ ϕ symmetric around 0, so

$= \Phi(\sqrt{x_2}) - \Phi(\sqrt{x_1}) - \Phi(\sqrt{x_1}) + \Phi(\sqrt{x_2}) = 2\Phi(\sqrt{x_2}) - 2\Phi(\sqrt{x_1})$

b) Using the result of a) and for $x_1=0$, $x_2=x$ we get

$P(x_1 \le X \le x_2) = 2\Phi(\sqrt{x}) - 2\Phi(0) = 2\Phi(\sqrt{x}) - 2\cdot\frac{1}{2} = 2\Phi(\sqrt{x}) - 1$

and by chain rule: $f_X(x) = \begin{cases} \frac{1}{\sqrt{x}}\varphi(\sqrt{x}) = \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{\pi}} e^{-x/2} = \frac{1}{\sqrt{2\pi x}} e^{-x/2}, & \text{for } x \ge 0 \\ 0 \text{ (because } x \ge 0 \text{ always)} & \text{otherwise} \end{cases}$

Question 3

a) $E(K/N=n) = \sum_{k} k\, P_{K/N}(k/n)$

$P_{K/N}(k/n) = \binom{n}{k} p^k (1-p)^{n-k} \longrightarrow$ so binomial (n,p)

and because of the information we have about the expectation of a binomial: $E(K/N=n) = np$

b) Now we call $\pi(n) = E(K/N=n)$,

so $E(K/N) = \pi(N) = Np$

and following $E(K) = E[\pi(N)] = E[Np] = pE(N) = p\lambda$

because N follows Poisson distribution with parameter λ

Question 4

$X \sim Bin\left(1500, \frac{1}{2}\right)$

and now applying the Central Limit Theorem we get

$$P(X \geq 950) \underset{\uparrow}{=} P(X \geq 949.5) = P\left(\frac{X-np}{\sqrt{np(1-p)}} \geq \frac{949.5-np}{\sqrt{np(1-p)}}\right) \simeq$$

continuity correction

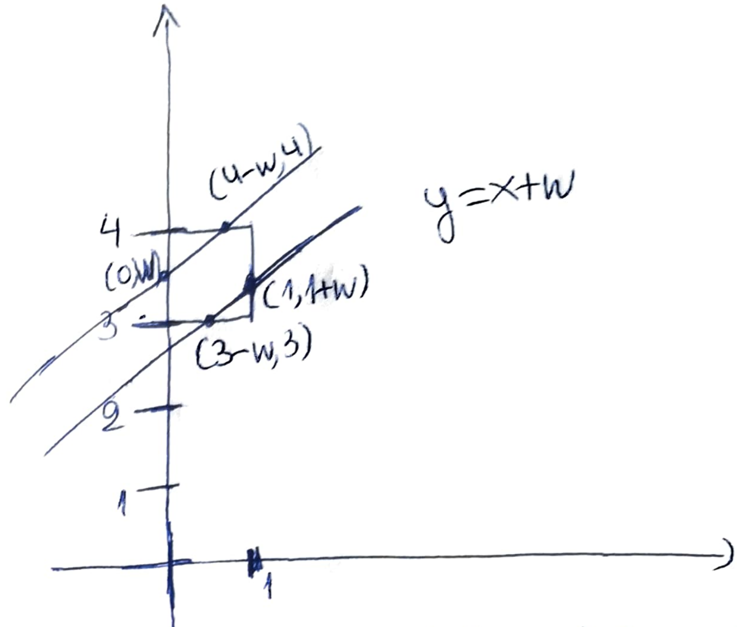
$$\simeq 1 - \Phi\left(\frac{949.5-750}{\sqrt{375}}\right) = 1 - \Phi\left(\frac{199.5}{\sqrt{375}}\right) \simeq 1 - \Phi(10.302)$$

Question 5

a) X and Y independent

so $f_{X,Y}(x,y) = f_X(x)\cdot f_Y(y)$

$$f_{X,Y}(x,y) = \begin{cases} 1, & 0 \le x \le 1,\ 3 \le y \le 4 \\ 0, & \text{else} \end{cases}$$



Calculating the areas on the graph we get the following cdf

$$F_W(w) = \begin{cases} 0, & w < 2 \\ \frac{1}{2}(w-2)^2, & 2 \le w < 3 \\ 1 - \frac{(4-w)^2}{2}, & 3 \le w \le 4 \\ 1, & w > 4 \end{cases}$$

area $\Delta \to 2 < w < 3$: $F_W(w) = \text{Area} = \frac{1}{2}(1+w-3)\cdot(1-(3-w)) = \frac{1}{2}(w-2)(w-2) = \frac{1}{2}(w-2)^2$

area  $\to 3 < w < 4$: $F_W(w) = \text{Area} = 1 - \frac{1}{2}(4-w)(4-w) = 1 - \frac{(4-w)^2}{2}$

b) $Var(W) = E(W^2) - [E(W)]^2$

Differentiating the cdf we get: $f_W(w) = \begin{cases} 0, & w < 2 \\ w-2, & 2 \le w < 3 \\ 4-w, & 3 \le w \le 4 \\ 0, & w > 4 \end{cases}$

$E(W) = \int w f(w)\,dw = \int_2^3 w^2 - 2w\,dw + \int_3^4 4w - w^2\,dw =$

$= \left[\frac{w^3}{3} - w^2\right]_2^3 + \left[2w^2 - \frac{w^3}{3}\right]_3^4 = \frac{4}{3} + \frac{5}{3} = 3$

$E(W^2) = \int w^2 f(w)\,dw = \int_2^3 w^3 - 2w^2\,dw + \int_3^4 4w^2 - w^3\,dw =$

$= \left[\frac{w^4}{4} - \frac{2}{3}w^3\right]_2^3 + \left[\frac{4}{3}w^3 - \frac{w^4}{4}\right]_3^4 = \frac{43}{12} + \frac{67}{12} = \frac{110}{12}$

Hence, $Var(W) = \frac{110}{12} - 9 = \frac{1}{6}$

c) $Y = X + 3 \Rightarrow Y - X = 3$

$Var(Y-X) = Var(3) = 0$, because the variance of a constant is zero