Midterm 202~~0~~2

1) a) $P(X=1) = P(\text{all balls have the same colour})$

$$= 3\cdot\left(\frac{1}{3}\right)^3 = \frac{3}{27} = \frac{1}{9}.$$

$P(X=3) = P(\text{you see each colour exactly once})$

$$= 3!\left(\frac{1}{3}\right)^3 = \frac{6}{27} = \frac{2}{9}$$

(first ball can have any colour, then 2 options for the second and the last one is fixed)

or $\quad 1\cdot\frac{2}{3}\cdot\frac{1}{3} = \frac{2}{9}.$

$$P(X=2) = 1 - P(X=1) - P(X=3) = 1 - \frac{1}{9} - \frac{2}{9} = \frac{2}{3}.$$

Or directly via counting:
3 options for the colour that occurs twice, then 2 for the colour that occurs once and 3 options for the order
(e.g. 2 red and 1 white can be rrw, rwr, wrr)

So $P(X=2) = 3\cdot2\cdot3\cdot\left(\frac{1}{3}\right)^3 = \frac{2}{3}.$

$$E(X) = \sum_x x\,P(X=x)$$

$$= 1\cdot\frac{1}{9} + 2\cdot\frac{2}{3} + 3\cdot\frac{2}{9}$$

$$= \frac{19}{9} = 2\tfrac{1}{9}.$$

b) A : at least one red
B : at least one blue.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A) = 1 - P(A^c) = 1 - \left(\frac{2}{3}\right)^3 = \frac{19}{27}$$

$$P(B) = \frac{19}{27}$$

$P(A \cap B) = P(1 \text{ red and } 2 \text{ blue})$
$+ P(2 \text{ red and } 1 \text{ blue})$
$+ P(1 \text{ red}, 1 \text{ blue}, 1 \text{ yellow})$

$$= 3 \cdot \left(\frac{1}{3}\right)^3 + 3 \cdot \left(\frac{1}{3}\right)^3 + 3! \cdot \left(\frac{1}{3}\right)^3$$

$\binom{3}{1}$ or $\binom{3}{2}$

$$= \frac{12}{27}.$$

So $P(A \cup B) = \frac{19}{27} + \frac{19}{27} - \frac{12}{27} = \frac{26}{27}.$

Alternative (easier):

$P(A^c \cap B^c)$

$$P(A \cup B) = 1 - P((A \cup B)^c)$$
$$= 1 - P(\text{no red and no blue})$$
$$= 1 - P(\text{all yellow}) =$$
$$= 1 - \left(\frac{1}{3}\right)^3 = \frac{26}{27}.$$

c) $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$.

$P(A \cap B)$ can be found in b) if you took the first option,

$$P(A|B) = \frac{\frac{12}{27}}{\frac{19}{27}} = \frac{12}{19}$$

d) No. Mention that $P(A \cap B) \neq P(A)P(B)$ or that $P(A|B) \neq P(A)$.

2 a) Let T be the event that the chip is declared defective and S the event that the chip is sound.

Observe that $P(S) = 0{,}95$, $P(S^c) = 0{,}05$

$P(T^c | S) = 0{,}97$ $\quad P(T|S) = 0{,}03$

$P(T| S^c) = 0{,}95$ $\quad P(T^c | S^c) = 0{,}05$

$$P(S|T) = \frac{P(S \cap T)}{P(T)}$$

$$\overset{\text{Bayes}}{=} \frac{P(S)P(T|S)}{P(S)P(T|S) + P(S^c)P(T|S^c)}$$

$$= \frac{0{,}95 \cdot 0{,}03}{0{,}95 \cdot 0{,}03 + 0{,}05 \cdot 0{,}95}$$

$$= \frac{0{,}0285}{0{,}076} = 0{,}375$$

2 b) Probability that a chip is declared sound is $P(T^c) = 1 - P(T) = 1 - 0{,}076 = 0{,}924$ as computed in a) (denominator)

Or: $P(T^c) = P(S)P(T^c|S) + P(S^c)P(T^c|S^c)$
$$= 0{,}95 \cdot 0{,}97 + 0{,}05 \cdot 0{,}05$$
$$= 0{,}924$$

$X \sim Bin(10, 0{,}924)$

$$P(X=k) = \binom{10}{k}(0{,}924)^k \cdot (0{,}076)^{10-k}$$
$k = 0, \dots, 10.$

2c) Y is geometric with parameter $0{,}076$

$$P(Y=k) = (0{,}924)^{k-1} \cdot 0{,}076$$
$k = 1, 2, \dots$

$$E(Y) = \frac{1}{0{,}076} = 13{,}2.$$

3) a) $\int_{-1}^{1} t x^4 \, dx = \left[\frac{1}{5} t x^5 \right]_{-1}^{1}$

$= t\left(\frac{1}{5} - \left(-\frac{1}{5}\right)\right) = \frac{2}{5} t$

$\int_{-\infty}^{\infty} f(x) \, dx = 1$, so $t = \frac{5}{2}$

b) $F_X(x) = 0$ for $x < -1$

For $-1 \leq x \leq 1$ $F_X(x) = \int_{-1}^{x} \frac{5}{2} s^4 \, ds$

$= \frac{1}{2} s^5 \Big|_{-1}^{x} = \frac{1}{2} x^5 - \left(-\frac{1}{2}\right)$

$= \frac{1}{2} x^5 + \frac{1}{2}.$

For $x > 1$: $F_X(x) = 1.$

c) $Var(X) = E(X^2) - E(X)^2$

$E(X) = 0$ (symmetry) or

$E(X) = \int_{-1}^{1} x \cdot \frac{5}{2} x^4 \, dx = \left[\frac{5}{12} x^6 \right]_{-1}^{1} = 0$

$E(X^2) = \int_{-1}^{1} x^2 \cdot \frac{5}{2} x^4 \, dx =$

$$= \int_{-1}^{1} \frac{5}{2} x^6 \, dx = \left[\frac{5}{14} x^7 \right]_{-1}^{1} =$$

$$= \frac{5}{14} - \left(- \frac{5}{14} \right) = \frac{10}{14} = \frac{5}{7}$$

$$Var(X) = \frac{5}{7}$$

d) $$E(|X|) = \int_{-1}^{1} |x| \cdot \frac{5}{2} x^4 \, dx$$

$$= \int_{-1}^{0} -x \cdot \frac{5}{2} x^4 + \int_{0}^{1} x \cdot \frac{5}{2} x^4 \, dx$$

$$= \left[- \frac{5}{12} x^6 \right]_{-1}^{0} + \left[\frac{5}{12} x^6 \right]_{0}^{1}$$

$$= \frac{5}{12} + \frac{5}{12} = \frac{10}{12} = \frac{5}{6}$$

$\left(\text{or } E(|X|) = 2 \cdot \int_0^1 x \cdot \frac{5}{2} x^4 \right).$

4) $P(X > -1{,}5) = P\left(\frac{X-2}{\sigma} > \frac{-1{,}5-2}{\sigma}\right)$

$$= 1 - \Phi\left(\frac{-3{,}5}{\sigma}\right) = 0{,}975$$

$$\Phi\left(\frac{-3{,}5}{\sigma}\right) = 0{,}025$$

So $\Phi\left(\frac{3{,}5}{\sigma}\right) = 1 - 0{,}025 = 0{,}975$

table: $\frac{3{,}5}{\sigma} = 1{,}96$

$$\sigma = \frac{3{,}5}{1{,}96} \approx 1{,}79$$

$$\sigma^2 \approx 3{,}19.$$